Chapter 1

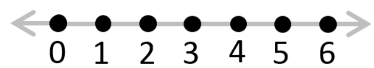
Operating on Rational Numbers

1.1 An Introduction to Operations

"Don't put the cart before the horse." is an old saying that makes the point you want to get things in the right order. In mathematics, at first, this isn't easy to do. For example, to discuss operators I'll need to use a mathematical expression and to discuss a mathematical expression, I'll need to refer to operators. My point is, it's going to take more than one reading to correctly begin organizing today's two important ideas, whole numbers and operators.

1.1.1 The Whole Numbers

In mathematics, a **set** is a well-defined and distinct collection of objects. We use braces, $\{\ \}$,to indicate a set and a comma to separate the objects. The set of objects $\{1,2,3,\ldots\}$ where the ellipsis, (the dots …) imply the numbers continue in the same pattern forever (in this case 4, 5, 6 etc.) is known as the set of **natural numbers**. If we also include 0 in the set, we have the **whole numbers**. (Although it won't affect our work, I want to mention that in some contexts people use the name natural numbers for the set we are calling the whole numbers.)

Numbers are often visualized using the "number line", 

, where the distance on the line from 0 to 1 is thought of as one "unit", the distance from 0 to 2 as two lengths of one unit and so on. The arrow to the right on the number line implies that 7, 8, 9 etc. will follow. For now, the line is only the "points" at 0, at 1, at 2 and so on. That's why the distances between the points are greyed out. In time, we'll "fill in" the rest of the distances by increasing the set of numbers we're allowed to work with.

Two whole numbers are considered "ordered" with the number to the right being "**greater than**" and the number to the left being "**less than**". For instance, 2 is less than 3 since 2 is to the left of 3 on the number line.

Although the whole numbers themselves are mathematically fascinating, we'll spend most of our time in this course combining numbers and operators.

1.1.2 Operations, Operators and Expressions

Operations transform numbers, and **operators** tells us which operation to perform. For instance, the addition operator, +, tells us to perform the operation of addition. When we see $5+15$ we know we're able to replace $5+15$ with 20. Today you'll only operate on numbers. Soon, you'll also operate on letters.

Numbers, letters and operators are used to build a mathematical **expression** which is a meaningful collection of numbers, letters, operations and the idea of grouping. (We'll look at grouping in a few minutes.) I'll use the word **simplify** when I want you to perform all the allowable operations in an expression. Before we begin simplifying expressions though, it's important that you and I share a common vocabulary for discussing operations.

1.1.3 Some Vocabulary for Operations

We add **terms** to get a **sum**. For example, in the expression 2 + 3, 2 is a term, 3 is a term and 5 is the sum. 2 + 3 is also a sum but the word "sum" usually refers to the final result.

We multiply **factors** to get a **product**. Three common ways to represent multiplication are $\times$ or $\bullet$ or ( )., so 2×5 or 2 • 5 or $2(5)$ or $(2)(5)$ all result in the product 10.

When we subtract a **subtrahend** from a **minuend** $(\text{minuend} - \text{subtrahend})$ we have a **difference**. For instance, with $5-3$, 5 is the minuend, 3 is the subtrahend and the difference would be 2. We use a "dash", $-$, to show the operation of subtraction.

We divide a **dividend** by a **divisor** to get a **quotient** and a **remainder**. To show the operation of division we'll use the $\div$ symbol ($\text{dividend} \div \text{divisor}$),a bar ($\frac{\text{dividend}}{\text{divisor}}$) or a slash, $\text{Dividend}/\text{Divisor}$. With today's problems, the remainder will always be 0 and remainders of 0 usually aren't written.

Mathematics is especially dependent on working (verbal) memory. To use your working memory well, you have to become confident with your vocabulary. A good way to memorize vocabulary is with a set of note cards. Write the word on the front and the definition and an example on the back. Words that are highlighted in **bold** in this textbook are good candidates for note cards. Every day use the cards to quiz yourself until you're automatic with the vocabulary of mathematics.

Before we begin the homework, I'd like to use our vocabulary to introduce the idea of a property. In mathematics a **property** allows us to use a general idea in specific situations. Please notice that with addition and multiplication we can use the same word for the number to the left and right of the operator but with subtraction and division we need different words for the number to the left and right of the operator. The idea that it doesn't make a difference which number is to the left of the operator and which is to the right is known as the **commutative property**.

***Property* – The Commutative Property of Addition**
The order of the terms doesn't affect the sum. Example: $3+2=2+3$ Note: Generally, subtraction is not commutative.

***Property* – The Commutative Property of Multiplication**
The order of the factors doesn't affect the product. Example: $3\times2=2\times3$ Note: Generally, division is not commutative.

Notice it's pronounced "commut ative" like commuting on a bus <u>not</u> "commun ative" like talking. Now, let's begin practicing with some vocabulary.

Practice 1.1.3 Some Vocabulary for Operations

Fill in the blanks using term, sum, factor, product, minuend, subtrahend, difference, dividend, divisor or quotient.

a) Before simplifying $\frac{4-2}{6+1}$;

4 is a(n) ____, 2 is a(n) ____, $4-2$ is a(n) ____ and a(n) ____, 6 is a(n) ____, 1 is a(n) ____, and $6+1$ is a(n) ____ and a(n) ____.

Minuend, subtrahend, difference, dividend, term, term, sum, divisor	$\Rightarrow$	Since 4 is to the left of the subtraction dash, it's the minuend. Since 2 is to the right of the subtraction dash, it's the subtrahend. $4-2$ is both a difference and, since it's above the division bar, it's the dividend. 6 is to the left of the addition symbol so it's a term. 1 is to the right of the addition symbol, so it's a term. $6+1$ is a sum and, since it's below the division bar, it's also the divisor.

Homework 1.1 Fill in the blanks using term, sum, factor, product, minuend, subtrahend, difference, dividend, divisor or quotient.

1) Before simplifying $2(3)$;
2 is a(n) ____, 3 is a(n) ____, and $2(3)$ is a(n) ____.

2) Before simplifying $20 \div 2$;
20 is a(n) ____, 2 is a(n) ____ and $20 \div 2$ is a(n) ____.

3) Before simplifying $18-12$;
18 is a(n) ____, 12 is a(n) ____, and $18-12$ is a(n) ____.

4) Before simplifying $\frac{7(2)}{4-3}$;
7 is a(n) ____, 2 is a(n) ____, $7(2)$ is a(n) ____ and a(n) ____, 4 is a(n) ____, 3 is a(n) ____ and $4-3$ is a(n) ____ and a(n) ____.

1.1.4 Ordering the Four Basic Operations

If I ask someone, who's good at algebra, the best name for $2(5-1)-4(3)$ they'll immediately answer, "It's a difference." That's because they've (usually unconsciously), ordered the four operators and realized the last operation they'll do is subtraction. Shifting the majority of your attention from numbers and letters to operations will help you take control of algebra.

To practice ordering operations, I'm going to ask you to count the number of operators and then name the operations using the correct order. Before we start though, I want to make a comment about PEMDAS (Parentheses, Exponents, Multiply, Divide, Add, Subtract) or other similar memory aids. Tools like PEMDAS are often too limited to handle problems in arithmetic or algebra, so if this is what you've been using, then it's time to expand the strength of your tools. For the next few weeks, you should have a printed copy of the full order of operations next to you and consciously consider each step as you simplify expressions.

***Procedure* – Order of Operations**
Begin with the innermost grouping idea and work out; Explicit grouping ( ), [ ], { } Implicit grouping Operations; in the dividend or divisor. inside absolute value bars. in radicands, exponents or logarithmic arguments. 1. Start to the left and work right simplifying each operation, different from the basic four, as you come to them. 2. Start again to the left and work right simplifying each multiplication or division as you come to them. 3. Start again to the left and work right simplifying each addition or subtraction as you come to them.

Before we start simplifying expressions, I want to mention that it's best to become automatic with the worked examples first and then use the homework to generalize what you've learned in the worked examples. Students who move to the homework first, and only look at the worked examples when they get stuck, often struggle as they try to find a pattern in one of the worked examples that will help them finish their problem. This type of pattern matching is a very weak strategy for "learning" algebra.

To get the most out of a worked example, copy the problem and, without looking at the textbook, write your own process and answer. Then look at the book and make sure both the answer <u>and the process</u> (the explanation to the right of $\Rightarrow$) match. If there wasn't a match, figure out what went wrong, (it's easy to do with a step by step worked example) copy the problem to a blank piece of paper, and try again. Only move to the next worked example when you're able to quickly finish your current worked example.

One last point, don't use a calculator when you're operating on whole numbers. You need to become automatic with the order of operations so the transition to letters will be easier.

Practice 1.1.4 Ordering the Four Basic Operations

Count the number of operators, name each operation using the correct order and then simplify the expression.

a) $1+2(3)+4$

There are three operators. Multiplication is first followed by the two additions which will be simplified left to right.

$1+6+4$	$\Rightarrow$	I started with step 2 in the general procedure and multiplied. Step 2 is finished.
$7+4$	$\Rightarrow$	I moved to step 3 of the general procedure, started again to the left and performed the first addition.
11	$\Rightarrow$	Continued moving to the right and performed the last addition.

b) $20 \div 2 - \frac{6}{2}$		
		There are three operators. The divisions are simplified first left to right and the subtraction is simplified last.
$10-3$	$\Rightarrow$	I simplified $20 \div 2$ and then $\frac{6}{2}$. Step 2 is finished.
7	$\Rightarrow$	I moved to step 3 and subtracted.
c) $14-10+8 \div 4(2)$		
		There are four operators. The multiplication and division are done first left to right. The addition and subtraction are done last left to right.
$14-10+2(2)$ $14-10+4$	$\Rightarrow$	Moving left to right, I divided first and then multiplied. Step 2 is finished.
$4+4$ 8	$\Rightarrow$	I began to the left again and subtracted first and added second. Step 3 is finished.

Homework 1.1 Count the number of operators, name each operation using the correct order and then simplify the expression.

5) $15-5(3)$	6) $\frac{6}{3}-1+1$	7) $4(5)+3(2)-2(4)$	8) $8(6)-8+6 \div 3$
9) $24-12 \div 6(2)$	10) $\frac{6}{3}\left(\frac{14}{2}\right)$	11) $18-\frac{14}{2}-7$	12) $80 \div 10(4)-2+2(2)$

1.1.5 Explicit Grouping

In the last topic, you saw that $1+2(3)$ simplifies to 7 (and not 9) because we multiply first and add second. If I wanted you to add first and multiply second, (so the result would be 9) I'd use an explicit grouping symbol like parentheses, $(1+2)(3)$ to force the addition to be first. With explicit grouping, you can see the grouping symbols. Parentheses ( ), brackets [ ], and braces { } are all examples of explicit grouping symbols.

Sometimes one kind of grouping symbol is **nested** within another. We often use different symbols for the inner and outer grouping so we can see what belongs together. To discuss $3[30-7(3+1)]$ we'd say, "The parentheses are nested inside the brackets."

One last point about grouping symbols. As we discussed earlier, a number written next to a grouping symbol implies multiplication. For instance, 7(4), $7[4]$, and $7\{4\}$ all simplify to 28.

Practice 1.1.5 Explicit Grouping

Count the number of operators, name the operations using the correct order and then simplify the expression.

a) $(6+3)-(6-3)$

There are three operators. The operations inside the parentheses are done first left to right and the subtraction is done last.

$9-(6-3)$	$\Rightarrow$	I added inside the left-most parentheses first. The first set of parentheses were no longer necessary.
$9-(3)$	$\Rightarrow$	I subtracted inside the next set of parentheses. I could have written $9-3$ but sometimes parentheses are kept even if they're not necessary.
6	$\Rightarrow$	I subtracted.

b) $12-\left(\frac{16}{4}+2\right)$

There are three operators. Inside the parentheses, division is first followed by the addition. The subtraction will be done last.

$12-(4+2)$	$\Rightarrow$	I divided inside the parentheses first.
$12-6$	$\Rightarrow$	Then I added inside the parentheses.
6	$\Rightarrow$	I subtracted last.

c) $3[30-7(3+1)]$

There are four operators. The addition inside the parentheses will be first, the multiplication inside the brackets will be next, the subtraction inside the brackets will follow and the multiplication outside the brackets will be last.

$3[30-7(4)]$	$\Rightarrow$	Since the parentheses are nested inside the brackets, I simplified the addition first.
$3[30-28]$	$\Rightarrow$	Inside the brackets I multiplied before I subtracted.
$3[2]$	$\Rightarrow$	I subtracted inside the brackets.
6	$\Rightarrow$	I finished by multiplying. (Remember $3[2]$ implies multiplication.)

Homework: 1.1 Count the number of operators, name each operation using the correct order and then simplify the expression.

13) $8+4-(8+4)$	14) $(15-8)(10-6)$	15) $4\{3(7-3)\}$
16) $36\div3(2+2)$	17) $8-2(5-3)$	18) $(1+3)[3(1+3)]$

1.1.6 Implicit Grouping

With implicit grouping we don't see a grouping symbols, but the idea of grouping still applies. One common use for implicit grouping involves quotients. To simplify a quotient, perform all the operations in the dividend (the top) and the divisor (the bottom) before you find the quotient and the remainder. For example, to simplify the quotient $\frac{18+6}{18-6}$ I'd first add in the dividend and subtract in the divisor, $\frac{24}{12}$, and only then get a quotient of 2.

Before simplifying this last set of problems, please take this advice to heart. Everyone does math, "In their head". People who do well with algebra keep a written record of each step they take. Only by keeping a step by step record of the process you thought was true can you review your work and learn from your mistakes.

Practice 1.1.6 Implicit Grouping

Count the number of operators, name each operation using the correct order and then simplify the expression.

a) $\frac{14-2}{2(4-2)}$

There are four operators. In the dividend subtract and, in the divisor, subtract first and multiply second. Finish by dividing.

$\frac{12}{2(2)}$	$\Rightarrow$	I subtracted in the dividend and then in the divisor.
$\frac{12}{4}$	$\Rightarrow$	I multiplied in the divisor.
3	$\Rightarrow$	To finish I found the quotient.

Homework 1.1 Count the number of operators, name each operation using the correct order and then simplify the expression.

19) $\frac{6-1}{2+3}$ 20) $\frac{(26-12)(2)}{26-12(2)}$ 21) $\frac{2+(2)(3)}{8\div(4)(2)}$ 22) $\frac{8}{5(2)-10}$ 23) $\frac{12+4(3)}{12}-\frac{12-4(3)}{12}$

1.1.7 More Vocabulary for Operations

Now that you've practiced the order of operations, you're able to discuss expressions in more detail. For example, should $13(2)+17$ be called a product, because of the multiplication, or a sum, because of the addition? The answer is, $13(2)+17$ is a sum, because, if you follow the order of operations, the <u>last</u> operation you'd perform would be the addition.

One last point, if an expression can be described using two different words, the order you choose to list those words, isn't important.

Practice 1.1.7 More Vocabulary for Operations

Fill in the blanks using term, sum, factor, product, minuend, subtrahend, difference, dividend, divisor or quotient.

a) Before simplifying $(3+4)(11-8)$;

3 is a(n) ____, 4 is a(n) ____, $3+4$ is a(n) ____, $(3+4)$ is a(n) ____, 11 is a(n) ____, 8 is a(n) ____, $11-8$ is a(n) ____, $(11-8)$ is a(n) ____, and $(3+4)(11-8)$ is a(n) ____.

term, term, sum, factor, minuend, subtrahend, difference, factor, product	$\Rightarrow$	Since 3 and 4 are added each is a term and $3+4$ is a sum. With parentheses, $(3+4)$, we have a factor. 11 is to the left of the dash so it's the minuend, 8 is to the right so it's the subtrahend and $11-8$ is a difference. With parentheses, $(11-8)$, the difference becomes a factor and so the entire expression, $(3+4)(11-8)$, is a product.

Homework 1.1 Fill in the blanks using term, sum, factor, product, minuend, subtrahend, difference, dividend, divisor or quotient.

24) Before simplifying $\frac{7+4}{5-2}$;

7 is a(n) ____, 4 is a(n) ____, $7+4$ is a(n) ____ and a(n) ____, 5 is a(n) ____, 2 is a(n) ____, $5-2$ is a(n) ____ and a(n) ____, and $\frac{7+4}{5-2}$ is a(n) ____.

25) Before simplifying $3(8)+2(5)$;

3 is a(n) ____, 8 is a(n) ____, $3(8)$ is a(n) ____ and a(n) ____, 2 is a(n) ____, 5 is a(n) ____, $2(5)$ is a(n) ____ and a(n) ____ and $3(8)+2(5)$ is a(n) ____.

26) Before simplifying $\frac{2(1+7)}{5+3}$;

2 is a(n) __, 1 is a(n) __, 7 is a(n) ___, $1+7$ is a(n) ___, $(1+7)$ is a(n) ___ and $2(1+7)$ is a(n) ___ and a(n) ___. 5 is a(n) ___, 3 is a(n) ___, $5+3$ is both a(n) ___ and a(n) ___ and $\frac{2(1+7)}{5+3}$ is a(n) ___.

27) Before simplifying $\left(6/1\right)\left(4/3\right)$;

6 is a(n) ___, 1 is a(n)___, $6/1$ is a(n) ____, $\left(6/1\right)$ is a(n) ____, 4 is a(n) ____, 3 is a(n) ____, $4/3$ is a(n) ____, $\left(4/3\right)$ is a(n) ____ and $\left(6/1\right)\left(4/3\right)$ is a(n) ____.

28) Before simplifying $2(15-1)+3(6+4)$;

2 is a(n) ____, 15 is a(n) ____, 1 is a(n) ____, $15-1$ is a(n) ____, $(15-1)$ is a(n) ____, $2(15-1)$ is a(n) ____ and a(n) ____, 3 is a(n) ____, 6 is a(n) ____, 4 is a(n)____, $6+4$ is a(n) ____, $(6+4)$ is a(n) ____, $3(6+4)$ is a(n) ____ and a(n) ____, and $2(15-1)+3(6+4)$ is a(n) ____.

29) Before simplifying $25-4(3)(2)$;

25 is a(n) ____, 4 is a(n) ____, 3 is a(n) ____, 2 is a(n) ____, $4(3)(2)$ is a(n) ____ and a(n) ____, and $25-4(3)(2)$ is a(n) ____.

30) Before simplifying $7+\frac{6-3(2)}{4}$;

7 is a(n) ____, 6 is a(n)____, 3 is a(n) ____, 2 is a(n) ____, $3(2)$ is a(n) ____ and a(n) ____, $6-3(2)$ is a(n)____ and a(n) ____, 4 is a(n)____, $\frac{6-3(2)}{4}$ is a(n) ____ and a(n) ____ and $7+\frac{6-3(2)}{4}$ is a(n) ____.

Homework 1.1 Answers

1) factor, factor, product   2) dividend, divisor, quotient   3) minuend, subtrahend, difference

4) factor, factor, product, dividend, minuend, subtrahend, difference, divisor

5) There are two operations. Multiplication is first followed by subtraction. The answer is 0.

6) There are three operations. The division is simplified first followed by the subtraction and then the addition. The answer is 2.

7) There are five operations. The multiplications are simplified first left to right, the addition is next followed by the subtraction. The answer is 18.

8) There are four operations. Multiplication is first, division is next and then we subtract and add left to right. The answer is 42.

9) There are three operations. The division and multiplication are simplified left to right followed by the subtraction. The answer is 20.

10) Currently we think of this as three operations. The correct order of operations is to divide the 6 by the 3, multiply the result by 14 and finally divide by 2. You'll often see people divide left to right first and multiply last. As we'll see later, the results will be the same. The answer is 14.

11) There are three operations. Division is first followed by subtracting left to right. The answer is 4.

12) There are five operations. The division is first followed by the multiplications left to right. The subtraction is next and the addition last. The answer is 34.

13) There are three operations. First add inside the parentheses, then add and subtract left to right. The answer is 0.

14) There are three operations. Simplify inside each set of parentheses left to right and finally multiply. The answer is 28.

15) There are three operations. The subtraction is first followed by multiplication by 3 and finally multiplication by 4. The answer is 48.

16) There are three operations. The addition is first, the division is next and the multiplication is last. The answer is 48.

17) There are three operations. The subtraction inside the parentheses is first, the multiplication by 2 is next and the subtraction is last. The answer is 4.

18) There are four operations. The additions inside parentheses are done first left to right. Then the multiplication by 3 inside the brackets is followed by the last multiplication. The answer is 48.

19) There are three operations. First, subtract in the dividend and add in the divisor, then divide. The answer is 1.

20) There are five operations. In the dividend subtract and multiply, in the divisor multiply and subtract. Last, divide the results. The answer is 14.

21) There are five operations. In the dividend multiply first and then add. In the divisor divide and multiply left to right. Finally divide. The answer is 2.

22) There are three operations. In the divisor multiply and subtract, then divide 8 by the result. After simplifying, the divisor becomes 0. The expression is undefined.

23) There are seven operations. For the first quotient multiply and add in the dividend, then divide. For the second quotient multiply and subtract in the dividend, then divide. Last, subtract the quotients. The answer is 2.

24) term, term, sum, dividend, minuend, subtrahend, difference, divisor, quotient

25) factor, factor, product, term, factor, factor, product, term, sum

26) factor, term, term, sum, factor, product, dividend, term, term, sum, divisor, quotient

27) dividend, divisor, quotient, factor, dividend, divisor, quotient, factor, product

28) factor, minuend, subtrahend, difference, factor, product, term, factor, term, term, sum, factor, product, term, sum

29) minuend, factor, factor, factor, product, subtrahend, difference

30) term, minuend, factor, factor, product, subtrahend, difference, dividend, divisor, quotient, term, sum

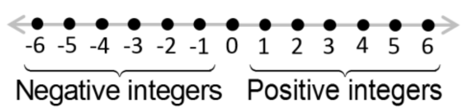
1.2 Multiplying and Dividing Integers

Over time, for several reasons, people found a need to move beyond the whole numbers. Merchants for example, realized the natural numbers easily counted the six sheep they had loaded on a ship, but, if the ship sank, the natural numbers couldn't count the loss of six sheep. To describe this loss, some merchants began to include a dash, $-$, along with the number. A natural number without the dash meant something "positive" (I have 6 sheep to sell), while the same number with a dash meant something "negative" (I've lost 6 sheep that I could have sold). In this section, we'll begin working with the integers, which allows us to count both positive and negative quantities.

1.2.1 The Integers

To build the set of **integers** we'll start with the natural numbers but rename them the, "**positive integers**". Next, we'll build the "**negative integers**" by preceding each positive integer with a dash. Last, we include 0. When you're considering whether a number is positive or negative you're considering the **sign** of the number. Zero is considered neither positive nor negative.

***Definition* – Integers**
The set of positive integers, negative integers and 0. Example $\{...-3, -2, -1, 0, 1, 2, 3...\}$

Visually, the integers number line runs left to right from "negative infinity" (since there are an infinite number of negative integers) to "positive infinity" (since there are an infinite number of positive integers), 

. The negative integers are to the left of zero and the positive integers are to the right of zero. Notice our points are still one unit apart so we can discuss a distance of 4 units or 5 units, but we can't yet discuss a distance that's between 4 and 5 units.

1.2.2 A Note About the Dash

When you operate on integers, you'll need to wrestle with the different ways we might interpret a dash. In the previous section, we used the dash to imply the operation of subtraction. Now we're using the same symbol to imply a negative number. In hindsight, using the same symbol for subtraction and sign might not have been the best choice, but it's what we did. As you'll see later, a couple other common uses for the dash is as an implicit factor of -1 and to represent the idea of opposite (negation).

1.2.3 Separating Size and Sign

When I interview people who consistently process signs correctly, they have an unconscious (automatic) process and a second conscious process which is based on the idea of separating issues of size from issues of sign. For instance, if I ask them to find the product $(-2)(-3)$, they'll automatically answer, "Six". If I ask them how they know the product is six most will say something like, "Well two times three is six and, since the product of two negative numbers is positive, my answer's positive six." Notice they begin by assuming both integers are positive. This is an example of finding a number's absolute value.

1.2.4 A Quick Look at Absolute Value

For now, we'll think of a number's absolute value as it's distance from 0 on the number line. So both positive five and negative five would have an absolute value of 5 since they're both 5 units from 0. Notice the absolute value of a non-zero number will always be positive.

We use two vertical bars, $|\ |$, as the operator for the operation of absolute value so both $|-5|$ and $|5|$ would simplify to 5. I don't want to spend much time discussing absolute value right now for a couple reasons. First, people who are good at processing signed numbers don't physically simplify absolute value as part of their procedure. Instead, they simply begin by assuming both numbers are positive or zero. Second, working with the mathematical definition of absolute value would take us too far afield today.

1.2.5 Multiplying Integers Automatically

Here's a procedure to help you multiply two integers automatically.

***Procedure* – Multiplying Two Non-Zero Integers**
1. Treat both integers as positive and find their product. 2. If both integers had the same sign, the product is positive. If they originally had different signs, the product is negative.

Notice our procedure doesn't address factors of 0. For that situation, there's a separate rule;

The Zero-Product Rule
The product of an integer and 0 can be replaced by 0. Example: $(0)(6) = 0$

Before moving on to the homework, please take this idea to heart. You'll only become automatic with signed expressions from repeated (correct) practice using your brain, not your calculator. In a while you'll need to automatically process sign issues as you <u>quickly</u> simplify expressions like $(x-5)(-x+2)$ where you won't be able to use a calculator.

Practice 1.2.5 Multiplying Integers Automatically *Simplify using the procedure.*		
a) $(-3)(5)$		
$(3)(5)=15$	$\Rightarrow$	I thought of both integers as positive and found their product.
-15	$\Rightarrow$	The final product is negative since the original integers had different signs.
b) $-7(2)(-2)$		
$-14(-2)$	$\Rightarrow$	I began multiplying left to right. $7(2)$ is 14 and since the signs are different the product is -14.
28	$\Rightarrow$	I continued to multiply. $14(2)$ is 28 and the product is positive since the signs were the same.

Homework 1.2 Simplify using the procedure.			
1) $(7)(-2)$	2) $-12(-10)$	3) $-2(-4)(-3)$	4) $(-3)(3)(-2)$
5) $(-3)(4)(-2)(-5)$	6) $-2((5)(-4))$	7) $-1[(-2)(7)(-3)]$	8) $-3\{-1[-2(5)]\}$

1.2.6 Dividing Integers

Notice the procedure for dividing integers mirrors the procedure for multiplying integers.

***Procedure* – Dividing Two Non-Zero Integers**
1. Treat both integers as positive and find their quotient.   2. If both integers had the same sign, the quotient is positive, if they had different signs, the quotient is negative.

What about 0? A quotient with a dividend of 0, and a divisor that isn't 0, is 0. For example, $\frac{0}{8}=0$. A quotient like $\frac{8}{0}$, with a divisor of 0 and a non-zero dividend, is undefined.

One final point. To simplify an expression like $-\frac{18}{9}$, where the dash is "in front" of the quotient, we'll process the quotient and then think of the dash as "opposite". That is, we'll think of it like this, $-\frac{18}{9}=-\left(\frac{18}{9}\right)=-(2)=-2$, where the opposite of $\frac{18}{9}$ is negative two.

Practice 1.2.6 Dividing Integers *Simplify.*		
a) $\frac{-18}{3}$		
-6	$\Rightarrow$	I thought of both 18 and 3 as positive and divided 18 by 3 to get 6. I ended with -6 since the signs of the divisor and dividend were different.

b) $-\,{}^{14}/_{-7}$	
$-(-2)$ $\Rightarrow$	First, I found the quotient of 14 and -7. 14 divided by 7 is 2 and the quotient is negative because the signs are different.
2 $\Rightarrow$	The opposite of negative two is positive two.

Homework 1.2 Simplify.			
9) $\frac{25}{-5}$	10) $-\,{}^{14}/_{7}$	11) $\frac{-18}{-9}$	12) $-\frac{-15}{3}$
13) $-\frac{-12}{-4}$	14) $-\frac{0}{5}$	15) $\frac{-(-30)}{-3}$	16) $-\,{}^{-21}/_{-(-7)}$

1.2.7 Combining Operations

Let's end this section with some problems that combine operations.

Practice 1.2.7 Combining Operations *Simplify.*	
a) $\frac{(-2)(-10)}{-4}$	
$\frac{20}{-4}$ $\Rightarrow$	Implicit grouping implies I should simplify the dividend first. The product of 2 and 10 is 20 and it's positive since the signs were the same.
-5 $\Rightarrow$	I divided 20 by 4. The quotient is negative since the signs were different.
b) $-\left(-{}^{30}/_{6}\right)\left({}^{15}/_{-3}\right)$	
$-(-5)\left({}^{15}/_{-3}\right)$ $\Rightarrow$	I simplified the first explicit grouping.
$-(-5)(-5)$ $\Rightarrow$	Then I simplified the second explicit grouping.
$(5)(-5)$ -25 $\Rightarrow$	Working left to right the opposite of negative five is positive five and then I multiplied to find the final product.
c) $-\frac{16-12}{12-10}$	
$-{}^{4}/_{2}$ $\Rightarrow$	I simplified the implicit groupings in the dividend and divisor first.
$-(2)$ -2 $\Rightarrow$	I found the quotient next. The opposite of positive two is negative two.

Homework 1.2 Simplify.			
17) $\frac{-((12)(-12))}{24}$	18) $-\left(\frac{-14}{2}\right)\left(\frac{14}{-2}\right)$	19) $\frac{(6)(-4)}{6-4}$	20) $-\frac{-2(-6)}{-4}$
21) $-\frac{28-14}{-14}$	22) $\frac{(-2)(-8)+8}{(4)(-2)}$	23) $\left(\frac{-9}{3}\right)\left(-\frac{9}{3}\right)\left(\frac{9}{-3}\right)$	24) $(-6)(-4)\div-2$
25) $-[-1(5)]+(-1)(-5)$	26) $-\frac{5+6}{-(18-7)}$		

Homework 1.2 Answers						
1) -14	2) 120	3) -24	4) 18	5) -120	6) 40	7) -42
8) -30	9) -5	10) -2	11) 2	12) 5	13) -3	14) 0
15) -10	16) 3	17) 6	18) -49	19) -12	20) 3	21) 1
22) -3	23) -27	24) -12	25) 10	26) 1		

1.3 Adding and Subtracting Integers

In this section, we'll continue operating on integers.

1.3.1 Adding Integers

Here's the procedure we'll use to add integers.

Procedure **– Adding Two Non-Zero Integers**
1. If the signs are the same, treat both integers as positive, find their sum and include the common sign. 2. If the signs are different, treat both integers as positive, subtract the smaller value from the larger and include the sign of whichever original integer had the larger absolute value.

Practice 1.3.1 Adding Integers

Simplify.

a) $-3+5$		
$5-3=2$	$\Rightarrow$	The original signs were different, so I thought of both integers as positive and subtracted the smaller value from the larger.
2	$\Rightarrow$	The sum is positive since 5 had the larger absolute value.
b) $-7+2+-3$		
$-7+2+-3$ $-5+-3$	$\Rightarrow$	Adding left to right the first two terms have different signs so I thought of both as positive and subtracted the smaller from the larger, $7-2=5$. The sign of the sum is negative since -7 had the larger absolute value.
-8	$\Rightarrow$	I continued to add. Since the signs are the same, I thought of both terms as positive, found their sum, $5+3=8$, and included the common sign.
c) $-3(5)+(-2)(-6)$		
$-15+12$	$\Rightarrow$	First, I multiplied left to right.
-3	$\Rightarrow$	Then, I finished with addition. Since the signs are different, I subtracted 12 from 15 to get three and ended with a negative three since the absolute value of -15 is greater than the absolute value of 12.

Homework 1.3 Simplify.

1) $12+-19$	2) $-45+82$	3) $-(-7+3)$	4) $-13+9+2$
5) $-8+2(-3)$	6) $-\left(\dfrac{-14}{2}\right)+\left(\dfrac{14}{-2}\right)$	7) $\dfrac{-7+-3}{-7+5}$	8) $(-6)(-2)+(-3)(5)$
9) $(-4)(15+-20)+8(2)$	10) $12+\dfrac{-12}{3}$	11) $(-6)(-4)+-2$	12) $-[-1(-5)]+(1)(-5)$
13) $\dfrac{-15+-4+12}{15+4+-12}$	14) $\dfrac{-15}{15}+-\dfrac{4}{4}+\dfrac{12}{-12}$	15) $\dfrac{5(-5)+(-3)(-3)}{(-5+3)(5+-3)}$	16) $-(7+-4)+-7+4$

1.3.2 Subtracting Integers

Earlier I mentioned that merchants started using negative numbers to represent the idea of a loss. For mathematicians, negative numbers helped extend the idea of a difference.

When students first encounter a subtraction problem like $5-3$, where the minuend is greater than the subtrahend, they're often asked to use their intuition with a story like this. "If you start with five apples, and take three apples away, you're left with two apples." That is, $5-3=2$. Unfortunately, this reasoning doesn't extend intuitively to a problem like $3-5$ where the minuend is less. Students are often uncomfortable if they're asked to, "Start with three apples and take away five apples." To simplify subtraction, we'll use a definition that remains consistent whether the minuend is greater than, less than, or equal to the subtrahend.

Definition **– Subtraction as Adding the Opposite**
The difference $a-b$ can be replaced with the sum of *a* and the opposite of *b*, $a+-b$. Example: $3-5$ can be replaced by $3+-5$

Here's a useful procedure that's based on the definition.

Procedure **– Subtracting Two Integers**
1. Identify the minuend and the subtrahend. 2. Change the subtraction to addition. 3. Change the subtrahend to its opposite. 4. Follow the procedure for adding two integers.

Practice 1.3.2 Subtracting Integers

Use the procedure to find the difference.

a) 9 – 17		
$9-17$	$\Rightarrow$	The minuend is 9 and the subtrahend is 17.
9 + –17	$\Rightarrow$	I changed the subtraction to addition and 17 to its opposite.
–8	$\Rightarrow$	Then I followed the procedure for addition.
b) –12 – 4		
–12 – 4	$\Rightarrow$	The minuend is -12 and the subtrahend is 4.
–12 + –4	$\Rightarrow$	I changed the subtraction to addition and 4 to its opposite.
–16	$\Rightarrow$	Then I followed the procedure for addition.
c) –7 – –3		
$-7--3$	$\Rightarrow$	The minuend is -7 and the subtrahend is -3.
$-7+3$	$\Rightarrow$	I changed the subtraction to addition and -3 to its opposite.
-4	$\Rightarrow$	Then I followed the procedure for addition.

Homework 1.3 *Use the procedure to find the difference.*			
17) $-8-10$	18) $5--2$	19) $4-11$	20) $-9--2$
21) $-10-4$	22) $1--6$	23) $-12--14$	24) $-13--13$

1.3.3 Combining Operations

It takes a lot of practice to become comfortable simplifying signs when subtraction is involved. For instance, when the class begins simplifying $5-4(2+1)$, it's common for a number of students to ask, "Is that a subtract four or a negative four?" The answer is, "It depends on your point of view."

If you take the point of view that the dash is a subtraction then, following the order of operations, you'd add first, multiply second and subtract last.

$$\begin{gathered} 5-4(2+1) \\ 5-4(3) \\ 5-12 \\ -7 \end{gathered}$$

On the other hand, if you decide to use the definition and change the subtraction to adding the opposite, then you're taking the point of view that it's a negative four. Notice that after simplifying your answer will again be -7.

$$\begin{gathered} 5-4(2+1) \\ 5+-4(2+1) \\ 5+-4(3) \\ 5+-12 \\ -7 \end{gathered}$$

With arithmetic, one of these approaches isn't "better" than the other. When it comes to algebra though, you'll want to take the point of view of adding the opposite (the second approach) so that's the procedure you should use for the homework. After finishing each of the following problems, I'd like you to check your answer using your calculator. Here's important information about using a calculator to simplify signed expressions.

Careful!	**The dash and your calculator**
	On most calculators it's the change sign key, usually (–) or ±, and not the subtraction key that changes the sign of a number. Also, on some calculators you enter the number first and then the sign while on others you enter the sign first and then the number.

Practice 1.3.3 Combining Operations

Working left to right identify whether each dash implies negative or subtraction. Next, change all subtractions to adding the opposite. Last, simplify the expression.

a) $-9--7$		
$-9--7$	$\Rightarrow$	Looking left to right the first dash implies negative, the second implies subtraction and the third implies negative.
$-9+7$	$\Rightarrow$	I changed the subtraction to adding the opposite.
-2	$\Rightarrow$	Then I simplified the addition.

b) $-4(-3)-3(5)$		
$-4(-3)-3(5)$	$\Rightarrow$	Left to right the first and second dash imply negative and the third dash implies subtraction.
$-4(-3)+-3(5)$	$\Rightarrow$	I changed the subtraction to adding the opposite.
$12+-15$ -3	$\Rightarrow$	I simplified using the order of operations.
c) $8-(4-11)$		
$8-(4-11)$	$\Rightarrow$	Both the first and second dash are subtractions.
$8-1(4-11)$	$\Rightarrow$	I made the factor of 1 after the subtraction explicit. Including this “unseen” factor of 1 at this point, either implicitly or explicitly, is common in algebra.
$8+-1(4+-11)$	$\Rightarrow$	I changed all subtractions to addition of the opposite.
$8+-1(-7)$	$\Rightarrow$	Since explicit grouping is next, I added inside the parentheses.
$8+7$	$\Rightarrow$	I multiplied next since, by the order of operations, multiplication comes before addition.
15	$\Rightarrow$	I finished by adding.

Homework 1.3 Working left to right identify whether each dash implies negative or subtraction. Next, change all subtractions to adding the opposite. Last, simplify the expression.

25) $-4-10-6$ 26) $(-2)-2(-7+3)$ 27) $-3(1-5)\div 6$ 28) $(-7)(2)-(-4)(3)$

29) $(-2)(-2)-2(3-7)$ 30) $\dfrac{-16}{4(1-(-3))}$ (Start with the dividend.)

31) $-(3-11)\div -2(1-5)$ 32) $7-[2-2(6-9)]$ 33) $-2-[-2-(2-6)]$

34) $7-5(2-(-8))$ 35) $-4[8-8(-1)-4]$

36) $\dfrac{2-14}{-4(5-8)}-\dfrac{36}{-4}$ (Start with the dividend.)

Homework 1.3 Answers

1) -7	2) 37	3) 4	4) -2	5) -14	6) 0
7) 5	8) -3	9) 36	10) 8	11) 22	12) -10
13) -1	14) -3	15) 4	16) -6		

17) The minuend is -8 and the subtrahend is 10. $-8+-10=-18$.

18) The minuend is 5 and the subtrahend is -2. $5+2=7$.

19) The minuend is 4 and the subtrahend is 11. $4+-11=-7$.

20) The minuend is -9 and the subtrahend is -2. $-9+2=-7$.

21) The minuend is -10 and the subtrahend is 4. $-10+-4=-14$.

22) The minuend is 1 and the subtrahend is -6. $1+6=7$.

23) The minuend is -12 and the subtrahend is -14. $-12+14=2$.

24) The minuend is -13 and the subtrahend is -13. $-13+13=0$.

25) The first dash implies negative, the second and third implies subtraction.
$-4+-10+-6=-20$

26) The first dash implies negative, the second implies subtraction and third implies negative.
$(-2)+-2(-7+3)=6$

27) The first dash implies negative and the second implies subtraction. $-3(1+-5)\div 6=2$

28) The first dash implies negative, the second dash implies subtraction and the third implies negative. The number following the subtraction is 1 (which, as usual, isn't written).
$(-7)(2)+-1(-4)(3)=-2$.

29) The first and second dash both imply negative, the third and fourth dash imply subtraction. The number following the first subtraction is 4 and the number following the second subtraction is 7. $(-2)(-2)+-2(3+-7)=12$.

30) The dash in the dividend implies negative. In the divisor, the first dash implies subtraction and the second, negative. The number following the subtraction is -3. $\dfrac{-16}{4(1+3)}=-1$.

31) The first dash implies negative (a factor of -1), the second implies subtraction the third implies negative and the fourth implies subtraction. The number following the first subtraction is 11 and the number following the second subtraction is 5.
$-(3+-11)\div -2(1+-5)=16$. (If you thought at first that the answer was 1, I did too when I was checking my answers.)

32) The three dashes all imply subtraction. The number following the first subtraction is 1, the number following the second subtraction is 2 and the number following the third subtraction is 9. $7+-1[2+-2(6+-9)]=-1$.

33) The first dash implies negative, the second implies subtraction the third implies negative and the fourth and fifth imply subtraction. The number following the first subtraction is 1 (which, as usual, isn't written). In the same way, the number following the second subtraction is also 1. The number following the third subtraction is a 6.
$-2+-1[-2+-1(2+-6)]=-4$.

34) The first and second dash imply subtraction, the third implies negative. The number following the first subtraction is 5. The number following the second subtraction is -8.
$7+-5(2+8)=-43$.

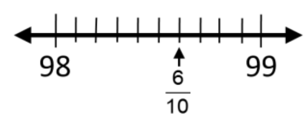
35) The first and third dash imply negative while the second and forth imply subtraction. The number following the first subtraction is 8 and the number following the second subtraction is 4. $-4[8+-8(-1)+-4]=-48$.

36) The dash in the dividend of the first term implies subtraction and, in the divisor, the left dash implies negative and the right dash implies subtraction. The dash between the rational expressions implies subtraction and the dash in the second term implies negative. The number following the first subtraction is 14. The number following the second subtraction is 8 and the number following the third subtraction is 1. $\dfrac{2+-14}{-4(5+-8)}+-1\left(\dfrac{36}{-4}\right)=8$.

1.4 Reducing Fractions

The integers allow us to count something “positive”, like profit and something “negative”, like debt. In both cases, though, we are counting units of 1 or -1. A nice visualization of this was the number line where the negative integers were counting units of -1 to the left of 0 and the positive integers were counting units of 1 to the right of 0. What if we wanted to describe a distance that ended up “between” two points? For that, we’ll need the idea of a rational number.

1.4.1 The Idea of a Rational Number



If your thermometer only had integers you could measure a temperature of 98° or 99° but you couldn’t measure a temperature of 98.6°. To measure the temperature 98.6° we divide the distance between 98 and 99 into 10 parts of equal distance and then use 6 of the 10 parts or $\frac{6}{10}$ parts. The number $\frac{6}{10}$ is an example of a rational number.

***Definition* – Rational Number**
A number that can be written as the quotient of two integers. Example: $3/5$, -0.1, 15% Note: Rational numbers with a dividend that isn’t 0 and a divisor that is 0 are undefined.

Fractions are a common type of rational number. In this section, we’ll begin reviewing operations with fractions. With a fraction the dividend is usually called the numerator and the divisor is usually called the denominator. Notice <u>the integers are a subset of the rational numbers</u> since every integer can be written as a fraction with a denominator of 1. For instance, the integer -3 can be written as the fraction $\frac{-3}{1}$.

1.4.2 Prime Factorization

Many procedures with fractions rely on prime factorization. To discuss prime factoring, we need a little vocabulary. A **prime number** is a natural number, greater than 1, which only has factors 1 and itself. The first few prime numbers are 2, 3, 5, 7, 11, 13, and 17. A **composite number** is a natural number, greater than 1, which is not prime. The first few composite numbers are 4, 6, 8, 9, 10, 12, and 14.

You have prime factored a number when the number is written as the product of only prime factors. We consider $2\times 2\times 3$, the **prime factorization** of 12 since the factors 2 and 3 are prime. (Here the $\times$ symbol means multiply, it’s not the letter *x*). We do not consider 2×6 a prime factorization of 12 because 6 is composite. To help organize a prime factorization it’s common to write the prime factors left to right from smallest to largest.

Practice 1.4.2 Prime Factorization

Prime factor any composite factors and use the commutative property to reorder the final prime factors from smallest to largest (left to right).

a) $3\times6\times2$		
$3\times2\times3\times2$	$\Rightarrow$	I prime factored 6. (The factorization $3\times3\times2\times2$ is also fine.)
$2\times2\times3\times3$	$\Rightarrow$	I reorder the factors smallest to largest left to right.

Homework 1.4 Prime factor any composite factors and use the commutative property to write the final prime factors from smallest to largest (left to right).

1) $5\times5\times15$	2) $4\times3\times6\times2$	3) $14\times9\times21$	4) $15\times6\times8$

1.4.3 Factor Rules

It's easy to see that the prime factorization of 6 is 2×3. It's harder to see that the prime factorization of 120 is $2\times2\times2\times3\times5$. The factor rules will help you decide if 2, 3 or 5 are factors of your number. You should be aware that it's probably more common to call the factor rules the divisibility rules.

Factor Rules for the Natural Numbers 2, 3 and 5
2 is a factor if the number is even. Even numbers end in 0,2,4,6 or 8.
3 is a factor if the sum of the number's digits is a multiple of 3.
5 is a factor if the number ends in 5 or 0.

Practice 1.4.3 Factor Rules

Decide if the following numbers have 2, 3, or 5 as factors.

a) 12,345		
2 is not a factor	$\Rightarrow$	Since the number does not end in 0,2,4,6 or 8 the number is not even.
3 is a factor	$\Rightarrow$	Since the sum of the digits is 15, $1+2+3+4+5=15$, and 15 has a factor of 3, 12,345 also has at least one factor of 3.
5 is a factor	$\Rightarrow$	12,345 ends in 5 so it has a factor of 5.
b) 120		
2 is a factor	$\Rightarrow$	Since the number ends in 0 it's even and has 2 as a factor.
3 is a factor	$\Rightarrow$	The sum of the digits is 3, $1+2+0=3$, so 3 is a factor.
5 is a factor	$\Rightarrow$	120 has a factor of 5 since it ends in 0.

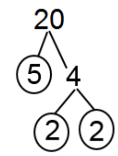
Homework 1.4 Decide if the following numbers have 2, 3, or 5 as factors.

5) 18	6) 540	7) 375	8) 119

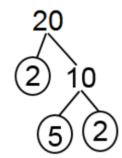
1.4.4 Factor Trees

Sometimes, a prime factorization will “pop” into your head. The factor rules, along with a factor tree, can help if nothing pops.

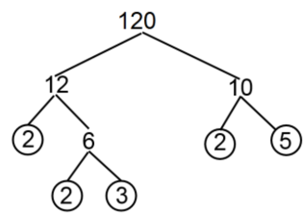
For example, to build a factor tree for 20, I might start with the factors 5 and 4. Since 5 is prime I’ll circle it and since 4 is composite I’ll continue to factor and then circle the 2’s since they’re prime. I can stop now since all the circled factors are prime. The prime factorization of 20 is $2 \times 2 \times 5$.



Notice that starting with 2 and 10 again leads to the prime factorization $2 \times 2 \times 5$. The idea that every composite number has a unique prime factorization, is often called the Fundamental Theorem of Arithmetic.



Here’s another example. When I see 120, I think of 12 times 10 so that’s how I’ll start. After continuing until only prime factors remain, I find that the prime factorization for 120 is $2 \times 2 \times 2 \times 3 \times 5$.



Sometimes finding factors of 2, 3 and 5 isn’t enough. For instance, 119 has prime factorization 7×17. Here’s a procedure that will help you find any prime factorization.

***Procedure* – To Prime Factor a Natural Number**
1. Build a factor tree using the factor rules for 2, 3, and 5.
2. After step 1 begin dividing uncircled factors by the prime numbers from 7 up to the natural number part of the number’s square root.
3. Write your prime factors in order from smallest to largest.

Homework 1.4 Prime factor the following numbers.					
9) 24	10) 42	11) 80	12) 91	13) 102	14) 429

1.4.5 Reducing Fractions

If the numerator and denominator of a fraction have a common factor, we can “**reduce**” the fraction using the fact that any non-zero number divided by itself is 1, and the fact that the product of a number and 1 can be replaced by the number itself.

For instance, to reduce $\frac{18}{30}$, I’d prime factor the numerator and denominator and write the prime factors from smallest to largest. Next, I’d “cross out” the common factor of 2 and a common factor of 3 since each is a factor of 1. After reducing, I’m left with three-fifths.

$$\frac{2 \times 3 \times 3}{2 \times 3 \times 5}$$

$$\frac{\cancel{2} \times \cancel{3} \times 3}{\cancel{2} \times \cancel{3} \times 5} = \frac{3}{5}$$

Notice that reducing "removes" unneeded factors of 1 from the product, it doesn't make the value of the fraction smaller. Here's a procedure to help reduce fractions.

***Procedure* – Reducing Fractions**
1. Prime factor the numerator and denominator.
2. Reduce factors common to both the numerator and denominator.
3. Find the product of any remaining factors in the numerator and then find the product of any remaining factors in the denominator.

Before you start on the homework, I want to caution you about an issue that comes up in class. Often, students reduce a fraction like $\frac{12}{18}$ using a process that starts like this, "Well two goes into twelve six times and two goes into eighteen nine times and then three goes into six twice and…" Although this idea can be used with arithmetic, it doesn't transfer well to algebra. For instance, later you'll be reducing $\frac{x^2-10x-24}{x^2-13x+12}$, and no one does this using the process, "Well ex minus twelve goes into ex squared minus ten ex minus twenty four ex plus two times…" On the other hand, prime factoring and reducing common factors transfers directly to algebra.

Practice 1.4.5 Reducing Fractions

Reduce using prime factorization.

a) $\frac{-36}{30}$		
$\frac{-1\times 2\times 2\times 3\times 3}{2\times 3\times 5}$	$\Rightarrow$	I prime factored the numerator and denominator. A common strategy in algebra is to think of -36 as -1×36.
$\frac{-1\times \cancel{2}\times 2\times \cancel{3}\times 3}{\cancel{2}\times \cancel{3}\times 5}$	$\Rightarrow$	I reduced common factors of 1.
$-\frac{6}{5}$	$\Rightarrow$	I finished by multiplying the unreduced factors in the numerator and then in the denominator.
b) $18\div 54$		
$\frac{2\times 3\times 3}{2\times 3\times 3\times 3}$	$\Rightarrow$	I prime factored the numerator and denominator.
$\frac{\cancel{2}\times \cancel{3}\times \cancel{3}}{\cancel{2}\times \cancel{3}\times \cancel{3}\times 3}=\frac{1}{3}$	$\Rightarrow$	I reduced and multiplied. I remembered to include a factor of 1 in the numerator (common factors are reduced to 1, not 0) since the remaining factor of 3 is in the denominator.
c) $^{-6}/_{180}$		
$\frac{-1\times \cancel{2}\times \cancel{3}}{2\times \cancel{2}\times \cancel{3}\times 3\times 5}=\frac{-1}{30}$	$\Rightarrow$	I prime factored the numerator and denominator and included a factor of -1 in the numerator. Next, I reduced common factors and found the remaining products.

Homework 1.4 Reduce using prime factorization.					
15) $\frac{52}{36}$	16) $-8 \div 40$	17) ${}^{42}\!/_{36}$	18) $63 \div 84$	19) $\frac{-56}{120}$	20) $\frac{105}{126}$
21) ${}^{28}\!/_{42}$	22) $-16 \div 20$	23) $\frac{102}{34}$	24) $84 \div -98$	25) ${}^{-105}\!/_{-126}$	
26) $\frac{45}{135}$	27) $\frac{143}{165}$	28) $-\frac{221}{-153}$			

Homework 1.4 Answers

1) $3\times5\times5\times5$ 2) $2\times2\times2\times2\times3\times3$ 3) $2\times3\times3\times3\times7\times7$ 4) $2\times2\times2\times2\times3\times3\times5$

5) 2 and 3 are factors. 6) 2, 3, and 5 are factors. 7) 3 and 5 are factors.

8) All three are **not** factors. 9) $2\times2\times2\times3$ 10) $2\times3\times7$ 11) $2\times2\times2\times2\times5$

12) 7×13 13) $2\times3\times17$ 14) $3\times11\times13$ 15) $\frac{2}{2}\times\frac{2}{2}\times\frac{13}{9}=\frac{13}{9}$

16) $\frac{2}{2}\times\frac{2}{2}\times\frac{2}{2}\times\frac{-1}{5}=-\frac{1}{5}$ 17) $\frac{2}{2}\times\frac{3}{3}\times\frac{7}{6}=\frac{7}{6}$ 18) $\frac{3}{3}\times\frac{7}{7}\times\frac{3}{4}=\frac{3}{4}$ 19) $\frac{2}{2}\times\frac{2}{2}\times\frac{2}{2}\times\frac{-7}{15}=\frac{-7}{15}$

20) $\frac{3}{3}\times\frac{7}{7}\times\frac{5}{6}=\frac{5}{6}$ 21) $\frac{\cancel{2}\times2\times\cancel{7}}{\cancel{2}\times3\times\cancel{7}}=\frac{2}{3}$ 22) $\frac{-1\times\cancel{2}\times\cancel{2}\times2\times2}{\cancel{2}\times\cancel{2}\times5}=-\frac{4}{5}$ 23) $\frac{\cancel{2}\times3\times\cancel{17}}{\cancel{2}\times\cancel{17}}=3$

24) $\frac{\cancel{2}\times2\times3\times\cancel{7}}{-1\times\cancel{2}\times7\times\cancel{7}}=-\frac{6}{7}$ 25) $\frac{\cancel{-1}\times\cancel{3}\times5\times\cancel{7}}{\cancel{-1}\times2\times\cancel{3}\times3\times\cancel{7}}=\frac{5}{6}$ 26) $\frac{\cancel{3}\times\cancel{3}\times\cancel{5}}{\cancel{3}\times\cancel{3}\times\cancel{5}\times3}=\frac{1}{3}$

27) $\frac{\cancel{11}\times13}{3\times5\times\cancel{11}}=\frac{13}{15}$ 28) $\cancel{-1}\times\frac{13\times\cancel{17}}{\cancel{-1}\times3\times3\times\cancel{17}}=\frac{13}{9}$

1.5 Multiplying and Dividing Fractions

In this section, we'll practice multiplying and dividing fractions.

1.5.1 Multiplying Fractions

Here's a fun way to multiply fractions using prime factorizations.

Procedure **– Multiplying Fractions**
1. Multiply the prime factored form of the numerators and denominators together. 2. Reduce common factors. 3. Multiply the remaining factors in the numerator and the remaining factors in the denominator.

Practice 1.5.1 Multiplying Fractions

Multiply using prime factorization.

a) $\left(\frac{6}{15}\right)\left(\frac{25}{28}\right)$

$\frac{2\times3\times5\times5}{3\times5\times2\times2\times7}$	$\Rightarrow$	I multiplied the prime factored form of the numerator and then of the denominator.
$\frac{\cancel{2}\times\cancel{3}\times\cancel{5}\times5}{\cancel{3}\times\cancel{5}\times\cancel{2}\times2\times7}=\frac{5}{14}$	$\Rightarrow$	I reduced the common factors and multiplied the remaining factors together.

b) $\frac{5}{9}\times\frac{-21}{15}\times3$

$\frac{5\times-1\times3\times7\times3}{3\times3\times3\times5}$	$\Rightarrow$	I thought of -21 as $-1\times3\times7$ and 3 as $3/1$ then I prime factored the numerator and denominator.
$\frac{\cancel{5}\times-1\times\cancel{3}\times7\times\cancel{3}}{\cancel{3}\times\cancel{3}\times3\times\cancel{5}}=-\frac{7}{3}$	$\Rightarrow$	I reduced the common factors and multiplied.

c) $2\frac{1}{3}\times-1\frac{2}{7}$

$\frac{7}{3}\times-\frac{9}{7}$	$\Rightarrow$	I wrote each mixed number as an improper fraction.
$\frac{\cancel{7}\times-1\times\cancel{3}\times3}{\cancel{3}\times\cancel{7}}=-3$	$\Rightarrow$	I multiplied, reduced common factors and multiplied the remaining factors.

Homework 1.5 Multiply using prime factorization.

1) $-\frac{3}{4}\times\frac{12}{15}$	2) $\left(\frac{24}{25}\right)\left(\frac{35}{30}\right)$	3) $\frac{-5}{24}\times18$	4) $7\times\frac{10}{21}\times-6$
5) $\left(2\frac{4}{7}\right)\left(1\frac{1}{27}\right)$	6) $\frac{-4}{9}\times\frac{6}{5}\times\frac{-15}{18}$	7) $\left(-3\frac{3}{4}\right)\left(\frac{1}{5}\right)\left(-1\frac{4}{9}\right)$	8) $2\frac{1}{7}\times\frac{14}{45}\times1\frac{1}{2}$
9) $\left(-\frac{21}{4}\right)\left(\frac{2}{-9}\right)\left(\frac{3}{5}\right)\left(\frac{-6}{15}\right)$	10) $\frac{13}{8}\times\frac{25}{49}\times4\times\frac{21}{39}$		

1.5.2 The Reciprocal

Before we begin dividing fractions, you'll need to know how to find a **reciprocal**. The reciprocal of a fraction is a second fraction which when multiplied to the first, gives a product of 1. For example, the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$ because $\left(\frac{2}{3}\right)\left(\frac{3}{2}\right)=\frac{6}{6}=1$. The reciprocal of -7 would be $-\frac{1}{7}$ since $\left(-\frac{7}{1}\right)\left(-\frac{1}{7}\right)=1$. Students often say that the reciprocal, "Is the original number flipped upside down." which is true, (as long as the original number isn't 0), but not very precise. Here's some practice finding a reciprocal.

Practice 1.5.2 The Reciprocal

Find the reciprocal of the given number.

a) $\frac{14}{15}$

$\frac{15}{14}$	$\Rightarrow$	Since $\left(\frac{14}{15}\right)\left(\frac{15}{14}\right)$ reduces to 1.

b) $-2\frac{3}{8}$

$-\frac{19}{8}$	$\Rightarrow$	I made sure to think of $-2\frac{3}{8}$ as $-1\times 2\frac{3}{8}$ before carrying out the procedure for rewriting as an improper fraction.
$-\frac{8}{19}$	$\Rightarrow$	The reciprocal is also negative since $\left(-\frac{19}{8}\right)\left(-\frac{8}{19}\right)=1$.

c) 0

Doesn't exist.	$\Rightarrow$	There isn't a number we can multiply to 0 to get a product of 1.

Homework 1.5 Find the reciprocal of the given number.

11) $\frac{40}{7}$ 12) $-\frac{5}{9}$ 13) $1\frac{3}{5}$ 14) -1 15) $-\frac{15}{14}$ 16) 8

1.5.3 Identifying the Numerator and Denominator of a Complex Fraction

You'll also need to be able to recognize the numerator and denominator of a complex fraction. A fraction, where the numerator, the denominator, or both are also fractions, is often called a **complex fraction**. An important skill when you're dividing complex fractions is being able to separate the numerator and denominator of the major fraction. For example, if we start with the complex fraction $\dfrac{\frac{2}{3}}{\frac{5}{6}}$ the numerator of the major fraction would be the minor fraction $\frac{2}{3}$

and the denominator would be the minor fraction $\frac{5}{6}$. Notice how the fraction bar separating the numerator and denominator of the major fraction is longer than the fraction bar for either of the minor fractions.

Sometimes, with a complex fraction like $\dfrac{-8}{\frac{40}{7}}$, students have a difficult time figuring out the numerator of the major fraction. If you put your attention on the larger fraction bar though, you see that the numerator is -8 and the denominator is $\frac{40}{7}$. It's common to rewrite the major fraction as $\dfrac{\frac{-8}{1}}{\frac{40}{7}}$ before beginning the procedure for dividing.

1.5.4 Dividing Fractions

Here's the procedure for dividing fractions.

Procedure **– Dividing Two Fractions**
To divide two fractions, multiply the minor fraction in the numerator by the reciprocal of the minor fraction in the denominator.

For instance, to simplify $\dfrac{\frac{2}{3}}{\frac{5}{6}}$, I'd recognize the numerator of the major fraction is $\frac{2}{3}$ and the denominator is $\frac{5}{6}$. Next, I'd multiply the numerator of the major fraction by the reciprocal of the denominator, $\left(\frac{2}{3}\right)\left(\frac{6}{5}\right)$. Last, I'd find the reduced product, $\left(\frac{2}{3}\right)\left(\frac{6}{5}\right) = \frac{2\times 2\times \cancel{3}}{\cancel{3}\times 5} = \frac{4}{5}$.

Practice 1.5.4 Dividing Fractions

After identifying the numerator and the denominator of the major fraction simplify using the procedure for dividing fractions.

a) $\dfrac{\frac{2}{5}}{\frac{4}{15}}$

The numerator is $\frac{2}{5}$ and the denominator is $\frac{4}{15}$.

$\left(\frac{2}{5}\right)\left(\frac{15}{4}\right)$	$\Rightarrow$	I multiplied the numerator by the reciprocal of the denominator.
$\frac{\cancel{2}\times 3\times \cancel{5}}{\cancel{5}\times \cancel{2}\times 2} = \frac{3}{2}$	$\Rightarrow$	I reduced common factors and multiplied.

b) $\frac{36}{5} \div \frac{4}{15}$		
The numerator is $\frac{36}{5}$ and the denominator is $\frac{4}{15}$.	$\Rightarrow$	The numerator is to the left of $\div$ and the denominator is to the right.
$\left(\frac{36}{5}\right)\left(\frac{15}{4}\right)$	$\Rightarrow$	I multiplied the numerator by the reciprocal of the denominator.
$\frac{\cancel{2}\times\cancel{2}\times 3\times 3\times 3\times \cancel{5}}{\cancel{5}\times\cancel{2}\times\cancel{2}} = 27$	$\Rightarrow$	Then I reduced common factors and multiplied.
c) $\frac{-2\frac{2}{3}}{4}$		
The numerator is $-2\frac{2}{3}$ and the denominator is 4.		
$\frac{-\frac{8}{3}}{\frac{4}{1}}$	$\Rightarrow$	I rewrote the numerator as an improper fraction and thought of the 4 in the denominator as $\frac{4}{1}$.
$\left(-\frac{8}{3}\right)\left(\frac{1}{4}\right) = -\frac{\cancel{2}\times\cancel{2}\times 2}{3\times\cancel{2}\times\cancel{2}} = -\frac{2}{3}$	$\Rightarrow$	I multiplied the numerator by the reciprocal of the denominator, reduced the two common factors of 2 and multiplied.

Homework 1.5 After identifying the numerator and the denominator of the major fraction simplify using the procedure for dividing fractions.

17) $\frac{\frac{8}{9}}{\frac{2}{3}}$ 18) $\frac{4}{21} \div \frac{8}{7}$ 19) $\frac{-8}{\frac{40}{7}}$ 20) $\frac{-1\frac{6}{7}}{3\frac{5}{7}}$ 21) $-\frac{\frac{9}{2}}{63}$

22) $\frac{4}{21} \div \frac{8}{7}$ 23) $\frac{14/15}{24/25}$ 24) $\frac{-4\frac{4}{5}}{-2\frac{2}{5}}$ 25) $-\frac{14}{3} \div \frac{7}{6}$ 26) $\frac{-\frac{15}{14}}{\frac{10}{21}}$

We'll finish with some problems where we combine multiplication and division.

Homework 1.5 Simplify.

27) $\frac{12}{20} \times \frac{7}{9} \div 14$ 28) $\frac{\frac{5}{12} \times \frac{9}{15}}{\frac{5}{9} \times \frac{6}{14}}$ 29) $\frac{3}{4} \div 1\frac{5}{9} \div \frac{-27}{4}$ 30) $\frac{8}{-13} \div -\frac{3}{26} \times \frac{-3}{5}$

Homework 1.5 Answers

1) $\dfrac{-1\times\cancel{3}\times\cancel{2}\times\cancel{2}\times 3}{\cancel{2}\times\cancel{2}\times\cancel{3}\times 5}=-\dfrac{3}{5}$ 2) $\dfrac{\cancel{2}\times 2\times 2\times\cancel{3}\times\cancel{5}\times 7}{\cancel{5}\times 5\times\cancel{2}\times\cancel{3}\times 5}=\dfrac{28}{25}$ 3) $\dfrac{-1\times 5\times\cancel{2}\times\cancel{3}\times 3}{\cancel{2}\times 2\times 2\times\cancel{3}}=-\dfrac{15}{4}$

4) $\dfrac{\cancel{7}\times 2\times 5\times -1\times 2\times\cancel{3}}{\cancel{7}\times\cancel{3}}=-20$ 5) $\dfrac{18}{7}\times\dfrac{28}{27}=\dfrac{\cancel{3}\times\cancel{3}\times 2\times 2\times 2\times\cancel{7}}{\cancel{7}\times\cancel{3}\times\cancel{3}\times 3}=\dfrac{8}{3}$

6) $\dfrac{\cancel{-1}\times\cancel{2}\times 2\times 2\times\cancel{3}\times\cancel{-1}\times\cancel{3}\times\cancel{5}}{\cancel{3}\times\cancel{3}\times\cancel{5}\times\cancel{2}\times 3\times 3}=\dfrac{4}{9}$ 7) $-\dfrac{15}{4}\times\dfrac{1}{5}\times-\dfrac{13}{9}=\dfrac{\cancel{-1}\times\cancel{3}\times\cancel{5}\times\cancel{-1}\times 13}{2\times 2\times\cancel{5}\times\cancel{3}\times 3}=\dfrac{13}{12}=1\dfrac{1}{12}$

8) $\dfrac{15}{7}\times\dfrac{14}{45}\times\dfrac{3}{2}=\dfrac{\cancel{3}\times\cancel{5}\times\cancel{2}\times\cancel{7}\times\cancel{3}}{\cancel{7}\times\cancel{3}\times\cancel{3}\times\cancel{5}\times\cancel{2}}=1$ 9) $\dfrac{\cancel{-1}\times\cancel{3}\times 7\times\cancel{2}\times\cancel{3}\times -1\times\cancel{2}\times\cancel{3}}{\cancel{2}\times\cancel{2}\times\cancel{-1}\times\cancel{3}\times\cancel{3}\times 5\times\cancel{3}\times 5}=\dfrac{-7}{25}$

10) $\dfrac{13}{8}\times\dfrac{25}{49}\times\dfrac{4}{1}\times\dfrac{21}{39}=\dfrac{\cancel{13}\times 5\times 5\times\cancel{2}\times\cancel{2}\times\cancel{3}\times\cancel{7}}{\cancel{2}\times\cancel{2}\times 2\times\cancel{7}\times 7\times\cancel{3}\times\cancel{13}}=\dfrac{25}{14}=1\dfrac{11}{14}$

11) $\dfrac{7}{40}$ 12) $-\dfrac{9}{5}$ 13) $\dfrac{5}{8}$ 14) -1 15) $-\dfrac{14}{15}$ 16) $\dfrac{1}{8}$

17) The numerator is $\dfrac{8}{9}$ and the denominator is $\dfrac{2}{3}$. The quotient is $\dfrac{4}{3}$

18) The numerator is $\dfrac{4}{21}$ and the denominator is $\dfrac{8}{7}$. The quotient is $\dfrac{1}{6}$

19) The numerator is -8 and the denominator is $\dfrac{40}{7}$. The quotient is $\dfrac{-7}{5}$

20) The numerator is $-\dfrac{13}{7}$ and the denominator is $\dfrac{26}{7}$. The quotient is $-\dfrac{1}{2}$

21) The numerator is $\dfrac{9}{2}$ and the denominator is 63. The quotient is $\dfrac{-1}{14}$

22) The numerator is $\dfrac{4}{21}$ and the denominator is $\dfrac{8}{7}$. The quotient is $\dfrac{1}{6}$

23) The numerator is $14/15$ and the denominator is $24/25$. The quotient is $\dfrac{35}{36}$

24) The numerator is $\dfrac{-24}{5}$ and the denominator is $\dfrac{-12}{5}$. The quotient is 2

25) The numerator is $-\dfrac{14}{3}$ or $\dfrac{14}{3}$ and the denominator is $\dfrac{7}{6}$. The quotient is -4

26) The numerator is $-\dfrac{15}{14}$ and the denominator is $\dfrac{10}{21}$. The quotient is $-\dfrac{9}{4}$

27) $\dfrac{1}{30}$ 28) $\dfrac{21}{20}$ 29) $-\dfrac{1}{14}$ 30) $-\dfrac{16}{5}$

1.6 Adding and Subtracting Fractions

Adding and subtracting fractions that don't share a common denominator is one of the more difficult topics in arithmetic. In this section we'll first add and subtract fractions that share a common denominator and then we'll work with fractions that don't share a common denominator.

1.6.1 Adding and Subtracting Fractions that have Common Denominators

For me, a fraction is a product where the denominator tells me the "size" of the fraction and the numerator counts how many I have. For example, I see $\frac{4}{7}$ as the product $\left(\frac{4}{1}\right)\left(\frac{1}{7}\right)$ which I'll write as $4\left(\frac{1}{7}\right)$. The first factor, the 4 (the original numerator), tells me I have four quantities size one-seventh $\left(\frac{1}{7}\right)$. So, when I see $\frac{4}{7}$, I visualize that I've cut the distance on the number line from 0 to 1 into seven equal pieces of size $\frac{1}{7}$ and I'm interested in the distance from 0 to the end of the fourth piece.

With this idea in mind it makes sense that to find the sum $\frac{4}{7}+\frac{3}{7}$ I should add the numerators and keep the common denominator since four one-sevenths and three more one-sevenths should be seven one-sevenths or seven sevenths, $\frac{4}{7}+\frac{3}{7}=\frac{4+3}{7}=\frac{7}{7}$. Notice how simplifying $\frac{7}{7}$ to 1 supports the idea that seven distances of size one-seventh moves us 1 unit to the right of 0 on the number line. Here's a procedure to add or subtract fractions that have a common denominator.

***Procedure* – Adding and Subtracting Fractions that have Common Denominators**
1. Add or subtract the numerators and keep the common denominator. 2. Reduce if possible.

Practice 1.6.1 Adding and Subtracting Fractions with Common Denominators *Simplify.*		
a) $\frac{4}{5}-\frac{7}{5}$		
$\frac{4-7}{5}$	$\Rightarrow$	Since the fractions have a common denominator, I subtracted the numerator values and kept the common denominator.
$\frac{-3}{5}$	$\Rightarrow$	I subtracted in the numerator.

b) $\frac{1}{3}+\frac{4}{3}-\frac{2}{3}$	
$\frac{1+4-2}{3}$ $\Rightarrow$	All the fractions have a common denominator, so I added and subtracted the numerators and kept the common denominator.
$\frac{5-2}{3}=\frac{3}{3}=1$ $\Rightarrow$	I simplified the numerator and then reduced the common factor.
c) $2\frac{1}{5}-3\frac{2}{5}$	
$\frac{11}{5}-\frac{17}{5}$ $\Rightarrow$	I rewrote the mixed numbers as improper fractions. (It's also possible to do this problem using mixed numbers and borrowing.)
$\frac{11-17}{5}=\frac{-6}{5}=-1\frac{1}{5}$ $\Rightarrow$	I subtracted in the numerator and kept the common denominator. Problems that begin with mixed numbers often end with mixed numbers.

Homework 1.6 Simplify.

1) $\frac{4}{15}+\frac{11}{15}$	2) $-\frac{7}{9}-\frac{11}{9}$	3) $5\frac{2}{3}+2\frac{1}{3}$	4) $\frac{7}{4}-\frac{5}{4}-\frac{-1}{4}$
5) $2\frac{4}{5}-\frac{1}{5}-1\frac{8}{5}$	6) $\frac{-1}{8}-\frac{9}{8}+\frac{5}{8}$	7) $\frac{-1}{11}-\frac{2}{11}-\frac{-3}{11}+\frac{4}{11}$	8) $\frac{32}{40}-\frac{15}{40}-\frac{28}{40}$

1.6.2 Building the LCD

Our procedure for adding and subtracting fractions works well when our fractions share a common denominator. What should we do with a problem like $\frac{2}{3}+\frac{1}{2}$ where the denominators are different? We'll replace any fraction without the common denominator with an "equivalent" fraction that has the common denominator. Then we'll use our previous procedure. When building a common denominator, we like to use the least (the smallest) common denominator which is usually referred to as the **LCD**. Here's a procedure for building the LCD.

***Procedure* – Building the LCD (Least Common Denominator)**
1. Prime factor each denominator. 2. Write down the prime factorization of the first denominator. This is the start of your LCD. 3. Now go through the remaining prime factorizations one by one and only include in your LCD the factors you need, but don't already have. 4. After considering all the denominators, multiply together the factors you included from steps 2 and 3. This is your LCD.

As you build the LCD I'd like you to practice a skill that will speed up your work when dealing with factors and products. The idea is based on the **associative property of multiplication**.

***Property* – The Associative Property of Multiplication**
The grouping of the factors doesn't affect the product. Example: $(2\times3)\times4=2\times(3\times4)$

To simplify a product like $2\times3\times3\times5$ "in my head", I'd use the commutative property to reorder the factors $2\times5\times3\times3$, use the associative property to regroup the factors $(2\times5)(3\times3)$ and then quickly find the product $(10)(9)$ or 90. With a factorization like $2\times7\times2\times3$ I notice if I reorder the factors, $7\times3\times2\times2$ and then regroup the factors, $(7\times3)\times2\times2$, that I'm doubling 21 to get 42, and then doubling 42 to get a final product of 84.

If this seems a bit much today, it's fine to continue finding the product using a calculator. If you do use a calculator though, please return to this topic in a few days and try the thought process again. As you'll find in this chapter, and in chapters 2 and 3, a lot of power comes from using the commutative and associative properties to quickly change your point of view when you find products from prime factorizations.

Practice 1.6.2 Building the LCD

Find the LCD of the following fractions.

a) $\frac{1}{12}$ and $\frac{5}{18}$		
$12=2\times2\times3$ $18=2\times3\times3$	$\Rightarrow$	I prime factored each denominator.
$2\times2\times3$	$\Rightarrow$	I copied the prime factorization of the first denominator so that 12 is included in the LCD.
$2\times2\times3\times3$	$\Rightarrow$	I moved on to 18 which has one factor of 2 and two factors of 3. Since the LCD already has one factor of 2 and one factor of 3, I only needed to include one more factor of 3 to have 18 in the LCD.
36	$\Rightarrow$	Since I've considered all the denominators, I grouped the factors as $(2\times2)(3\times3)$ and multiplied $(4)(9)$ to find the LCD is thirty-six. Another idea would be $(2\times3)(2\times3)=(6)(6)$ which again gives the product thirty-six.
b) $\frac{-9}{14}$ and $1\frac{5}{12}$ and $\frac{2}{21}$		
$14=2\times7$ $12=2\times2\times3$ $21=3\times7$	$\Rightarrow$	I prime factored each denominator.

2×7	$\Rightarrow$	I copied down the 2×7 so that 14 is part of the LCD.
$2\times7\times2\times3$	$\Rightarrow$	Next, I moved on to 12. The LCD already has one of my factors of 2 so I only need a second factor of 2 and a factor of 3 to have 12 included in my LCD.
$2\times7\times2\times3$	$\Rightarrow$	Now I moved on to the last denominator, (the 21) and realized the LCD already has a 21 so no new factor is needed.
84	$\Rightarrow$	I used $(7\times3)\times2\times2$ to see the LCD is 84.

c) $4\frac{7}{10}$ and $\frac{25}{9}$ and $\frac{2}{25}$		
$10=2\times5 \quad 9=3\times3 \quad 25=5\times5$	$\Rightarrow$	I prime factored the denominators.
2×5	$\Rightarrow$	I copied down the first denominator. 10 is now part of the LCD.
$2\times5\times3\times3$	$\Rightarrow$	I needed two factors of 3 to include a factor of 9 in the LCD.
$2\times5\times3\times3\times5$	$\Rightarrow$	I needed a second factor of 5 so that 25 is included in the LCD.
450	$\Rightarrow$	I used $(3\times3)(5)(2\times5)$ to see that nine times five is forty-five and forty-five times ten gives an LCD of 450.

Homework 1.6 Find the LCD of the following fractions.

9) $\frac{3}{4}, \frac{3}{10}$	10) $\frac{1}{12}, \frac{9}{8}$	11) $-\frac{9}{14}, -\frac{5}{6}$	12) $-\frac{1}{2}, \frac{4}{3}, \frac{5}{6}$	13) $-\frac{1}{6}, 2\frac{5}{16}, \frac{7}{12}$
14) $-2\frac{3}{4}, -\frac{5}{36}, \frac{13}{20}$	15) $\frac{5}{9}, \frac{7}{10}, \frac{1}{4}, \frac{2}{25}$	16) $-\frac{1}{6}, \frac{5}{18}, \frac{3}{4}, \frac{8}{15}$		

1.6.3 Building Equivalent Fractions

Once you've found the least common denominator, you'll need to replace each fraction without the LCD, with an **equivalent fraction** that has the LCD. Two fractions are equivalent if they have the same value.

We can change a fraction's look, without changing its value, if we multiply the original fraction by 1. The form of 1 we choose will depend on the factors in our LCD that are missing from our original denominator. For example, say I started with $\frac{4}{7}$ and knew the common denominator was 35. Then the form of 1 I'd want would be $\frac{5}{5}$ since a factor of 5 is needed to take my original denominator of 7 to my common denominator of 35, $\frac{4}{7}\times1=\frac{4}{7}\times\frac{5}{5}=\frac{20}{35}$. Please notice the original numerator also had to be multiplied by 5 so the final fraction was equivalent to the original. Here's a procedure to help build equivalent fractions.

Procedure **– Building Equivalent Fractions**
1. Find the factor, which when multiplied to the original denominator, gives the LCD as the product. (If you don't "see" the factor, you can always divide the LCD by the original denominator.) 2. Multiply both the numerator and denominator of the original fraction by the factor found in step 1.

Here's some practice building equivalent fractions.

Practice 1.6.3 Building Equivalent Fractions

Rewrite the factor of 1 and multiply, so the product is an equivalent fraction with the new denominator.

a) $\frac{4}{5}\times 1=\frac{\quad}{20}$

$\frac{4}{5}\times\frac{4}{4}=\frac{\quad}{20}$	$\Rightarrow$	Since a factor of 4 takes my original denominator (the 5) to my LCD (the 20) I multiplied by 1 in the form $\frac{4}{4}$.
$\frac{4}{5}\times\frac{4}{4}=\frac{16}{20}$	$\Rightarrow$	I multiplied the fractions. $\frac{16}{20}$ has the same value as $\frac{4}{5}$.

b) $-\frac{7}{8}\times 1=-\frac{\quad}{104}$

$-\frac{7}{8}\times\frac{13}{13}=-\frac{\quad}{104}$	$\Rightarrow$	I used a calculator to divide, $104\div 8=13$, and found that $\frac{13}{13}$ is the form of 1 I want.
$-\frac{7}{8}\times\frac{13}{13}=-\frac{91}{104}$	$\Rightarrow$	I multiplied to build the equivalent fraction.

Homework 1.6 Rewrite the factor of 1 and multiply, so the product is an equivalent fraction with the new denominator.

17) $\frac{5}{6}\times 1=\frac{\quad}{48}$ 18) $\frac{-3}{1}\times 1=\frac{\quad}{6}$ 19) $\frac{18}{25}\times 1=\frac{\quad}{75}$ 20) $\frac{11}{9}\times 1=\frac{\quad}{117}$ 21) $\frac{-2}{9}\times 1=\frac{\quad}{153}$

1.6.4 Adding and Subtracting Fractions

Now that you can find the LCD and build equivalent fractions, you're ready to add and subtract fractions that don't share a common denominator. Here's our general procedure.

Procedure **– Adding and Subtracting Fractions**
1. Find the LCD of all the denominators. 2. Replace each fraction without the LCD, with an equivalent fraction that has the LCD. 3. Add or subtract the numerators. Keep the common denominator. 4. Reduce if possible.

Practice 1.6.4 Adding and Subtracting Fractions

Simplify.

a) $\frac{1}{2}-\frac{1}{3}$

$2\times3=6$	$\Rightarrow$	I found the LCD.
$\frac{1}{2}=\frac{3}{6}$ and $\frac{1}{3}=\frac{2}{6}$	$\Rightarrow$	I built equivalent fractions using the LCD.
$\frac{3}{6}-\frac{2}{6}=\frac{3-2}{6}=\frac{1}{6}$	$\Rightarrow$	After replacing the original fractions with their equivalent fractions, I subtracted the numerators and kept the LCD.

b) $\frac{7}{6}-\frac{5}{14}+\frac{7}{12}$

$2\times2\times3\times7=84$	$\Rightarrow$	I found the LCD.
$\frac{7}{6}=\frac{98}{84}$ $\frac{5}{14}=\frac{30}{84}$ $\frac{7}{12}=\frac{49}{84}$	$\Rightarrow$	Then I built equivalent fractions using the LCD.
$\frac{98-30+49}{84}=\frac{117}{84}$	$\Rightarrow$	Next, I added and subtracted the numerators and kept the LCD.
$\frac{117}{84}=\frac{\cancel{3}\times3\times13}{2\times2\times\cancel{3}\times7}=\frac{39}{28}$	$\Rightarrow$	Since both 117 and 84 share a common factor of 3. (The sum of their digits are both a multiple of 3, $1+1+7=9$ and $8+4=12$.) I factored and reduced.

c) $1\frac{1}{6}-2\frac{5}{9}$

$\frac{7}{6}-\frac{23}{9}$	$\Rightarrow$	I find it's often easiest if I replace mixed numbers with their equivalent improper fraction.
$2\times3\times3=18$	$\Rightarrow$	I found the LCD.
$\frac{7}{6}=\frac{21}{18}$ $\frac{23}{9}=\frac{46}{18}$	$\Rightarrow$	Then I built equivalent fractions using the LCD.
$\frac{21-46}{18}=-\frac{25}{18}=-1\frac{7}{18}$	$\Rightarrow$	I subtracted the numerators, kept the LCD and wrote the difference as a mixed number.

Homework 1.6 Simplify.

22) $\frac{4}{15}+\frac{2}{5}$	23) $\frac{5}{7}-\frac{3}{14}$	24) $\frac{3}{4}+\frac{5}{16}$	25) $3\frac{1}{2}+1\frac{2}{3}$
26) $\frac{1}{9}-2$	27) $\frac{5}{8}-\frac{-1}{18}$	28) $3-\frac{7}{5}$	29) $\frac{1}{4}+4$
30) $\frac{7}{9}+1\frac{1}{12}$	31) $\frac{3}{10}-1\frac{1}{15}$	32) $-\frac{1}{2}+\frac{2}{3}-\frac{3}{4}$	33) $\frac{4}{5}-\frac{3}{8}-\frac{7}{10}$

34) $2\frac{1}{6}+3-1\frac{7}{9}$	35) $\frac{1}{7}-\frac{1}{5}+\frac{23}{35}$	36) $\frac{3}{10}-1\frac{1}{15}+\frac{6}{5}$	37) $\frac{-4}{9}-\frac{-5}{3}-\frac{5}{4}$
38) $6-1\frac{5}{12}-4\frac{7}{18}$	39) $\left(\frac{4}{5}\right)\left(\frac{4}{5}\right)-\frac{4}{5}$	40) $\frac{3}{4}-\frac{2}{3}\div\frac{4}{3}$	41) $\left(\frac{3}{4}+\frac{3}{2}\right)\div\frac{27}{10}$

Homework 1.6 Answers

1) $\frac{15}{15}=1$	2) $\frac{-18}{9}=-2$	3) $\frac{24}{3}=8$	4) $\frac{3}{4}$	5) $\frac{0}{5}=0$	6) $\frac{-5}{8}$	
7) $\frac{4}{11}$	8) $\frac{-11}{40}$	9) $2\times2\times5=20$	10) $2\times2\times2\times3=24$	11) $2\times3\times7=42$		
12) $2\times3=6$	13) $2\times2\times2\times2\times3=48$	14) $2\times2\times3\times3\times5=180$				
15) $2\times2\times3\times3\times5\times5=900$	16) $2\times2\times3\times3\times5=180$	17) $\frac{5}{6}\times\frac{8}{8}=\frac{40}{48}$				
18) $\frac{-3}{1}\times\frac{6}{6}=\frac{-18}{6}$	19) $\frac{18}{25}\times\frac{3}{3}=\frac{54}{75}$	20) $\frac{11}{9}\times\frac{13}{13}=\frac{143}{117}$	21) $\frac{-2}{9}\times\frac{17}{17}=-\frac{34}{153}$			
22) $\frac{2}{3}$	23) $\frac{1}{2}$	24) $\frac{17}{16}$	25) $5\frac{1}{6}$	26) $-\frac{17}{9}$	27) $\frac{49}{72}$	28) $\frac{8}{5}$
29) $\frac{17}{4}$	30) $1\frac{31}{36}$	31) $-\frac{23}{30}$	32) $-\frac{7}{12}$	33) $-\frac{11}{40}$	34) $3\frac{7}{18}$	35) $\frac{3}{5}$
36) $\frac{13}{30}$	37) $-\frac{1}{36}$	38) $\frac{7}{36}$	39) $\frac{-4}{25}$	40) $\frac{25}{36}$	41) $\frac{5}{6}$	

1.7 Applying the Order of Operations

Earlier we discussed that a mathematical expression is a meaningful collection of numbers, letters, operators and grouping. In this section, we're going to begin **evaluating** an expression by substituting a number in place of a letter and then simplifying.

1.7.1 A Word About Variables

Although it's common to call any letter that stands in place of an unknown number a "variable", it's important to appreciate that sometimes we expect the value will vary (will change) and sometimes we don't. For example, in the first topic below, we'll practice checking whether 4 solves the equation $x+2(x+1)=11$. In this context, the variable *x* is standing in place of a single unknown number, so once we chose a number, its value really isn't expected to change.

Next, we'll look at using formulas. A **formula** uses mathematics to express a rule or a fact. For example, the formula $F={}^{9}\!/_{5}C+32$ helps us convert a Celsius temperature (represented by *C* in the formula), to a Fahrenheit temperature (represented by *F* in the formula). Notice that as we vary the value of *C* then, after multiplying by $\frac{9}{5}$ and adding 32, the resulting value for *F* will also change. With a formula we do expect the values might vary.

1.7.2 Checking an Answer

When two expressions are separated by an **equality symbol**, =, we have an **equation**. An equation implies the expression to the left of the equality symbol and the expression to the right of the equality symbol have the same value. For instance, the equation $x+1=5$ implies the left side, $x+1$, and the right side, 5, are both 5. A **solution** for $x+1=5$ is any value that evaluates to 5 on the left. I hope you can see that 4 is a solution for $x+1=5$ since, after substituting 4 for *x* and simplifying, the left side also takes on the value 5. We have **solved** an equation when we find all the values that are a solution. These values are collected in a **solution set**. The solution set for $x+1=5$ would be $\{4\}$.

Unlike $x+1=5$, where most people can "see" the solution is 4, it's uncommon for someone to look at $-2y-1=-3(y+5)$ and have the solution -14 pop into their head. Later in our course, you'll learn a procedure that helps you find the solution for $-2y-1=-3(y+5)$. Unfortunately, if you make a mistake carrying out the procedure, your answer probably won't match the solution. To convince ourselves that our answer and the solution match, we'll return to the original equation and replace every variable with our answer. If, after simplifying, the left and right expressions are the same value, we assume that our answer is the solution. If instead, the left value doesn't match the right value, we assume we made a mistake carrying out the procedure. This process of evaluating an answer, is known as **checking**. Here's an example.

Say we started with $2(y-1)=y+4$ and, after carrying out the procedure, we came up with the answer 5. To check our answer, we'd return to the <u>original</u> equation, replace every *y* with 5 and simplify the left and right side to see if both expressions take on the same value. Unfortunately, in this case, the left side evaluates to 8 and the right side to 9. This tells us that our answer, 5, probably isn't the solution and we should go through our solution procedure, step by step, and look for errors.

$$\begin{array}{cc} 2(y-1) & y+4 \\ 2(5-1) & 5+4 \\ 2(4) & 9 \\ 8 & \end{array}$$

Let's assume we found an error, carried out the procedure a second time and found a new answer, 6. After evaluating each expression with 6, we find the left and right expression both take on the value 10. This implies that 6 is the solution and our solution set would be $\{6\}$.

$$\begin{array}{cc} 2(y-1) & y+4 \\ 2(6-1) & 6+4 \\ 2(5) & 10 \\ 10 & \end{array}$$

Please notice that I don't put an equality symbol between my expressions when I begin checking an answer. If I did, I'd be assuming the values are the same before I've shown the two expressions do have the same value. Here's some practice checking an answer.

Practice 1.7.2 Checking an Answer

Check the supplied answer and state whether it is, or is not, the solution.

a) $x+2(x+1)=11$ with answer 4

$4+2(4+1) \quad 11$	$\Rightarrow$	I substituted 4 for every *x* on the left side. The right side is already simplified to 11.
$\begin{array}{cc} 4+2(5) & 11 \\ 4+10 & \\ 14 & \end{array}$	$\Rightarrow$	Then I simplified the left side. Since the values, the 14 and the 11, are different, 4 is not the solution.

b) $-2y-1=-3(y+5)$ with answer -14

$-2(-14)-1 \quad -3(-14+5)$	$\Rightarrow$	I substituted -14 for every *y*.
$\begin{array}{cc} 28-1 & -3(-9) \\ 27 & 27 \end{array}$	$\Rightarrow$	I simplified each side using the order of operations. Since both expressions simplify to the same value, the 27, -14 is the solution.

c) $k/2-k/3=(k-1)/4$ with answer 3

$3/2-3/3 \quad (3-1)/4$	$\Rightarrow$	I substituted 3 for every *k*.
$\begin{array}{cc} 9/6-6/6 & 2/4 \\ 3/6 & \end{array}$	$\Rightarrow$	After simplifying, I see that both fractions can be reduced to one-half so 3 is the solution.

Homework 1.7 Check the supplied answer and state whether it is, or is not, the solution.	
1) $46 = 2L + 2(12)$ with answer 11	2) $2k - 5 = -9k - 27$ with answer -2
3) $-3(y+1) = 3(y+1)$ with answer 1	4) $\frac{2}{15} + \frac{a}{5} = \frac{1}{3}$ with answer 1
5) $\frac{x-2}{2} = \frac{x+4}{6}$ with answer 5	6) $\frac{y-2}{18} - \frac{y}{4} = \frac{-8}{9}$ with answer 4
7) $3(x-1) + 2 = x + 4$ with answer 2	8) $2(4-t) = 2t - (3t+4)$ with answer 12
9) $\frac{3+k}{9} - \frac{k}{6} = \frac{k}{36} + 2$ with answer -20	10) $-14 - 5(a+3) = -(1-2a)$ with answer -4
11) $-15 = -2(z+1) + 3(z-1)$ with answer -10	12) $x + \frac{x}{2} - \frac{1}{4} = \frac{x}{3} + \frac{1}{3}$ with answer -1

1.7.3 Applying Formulas - Midpoint

Sometimes, we'd like to know the value on the number line that's midway between two points. This value is an example of a midpoint. We can use the formula $M = \frac{x_1 + x_2}{2}$ to find the midpoint on a number line. Here M stands for the value of the midpoint and the x_1 and x_2 are the two points. Here's some practice finding the midpoint.

Practice 1.7.3 Applying Formulas - Midpoint *Find the midpoint for the following pair of values.*		
a) -11 and -1.		
$M = \frac{-11 + -1}{2}$	$\Rightarrow$	I replaced x_1 and x_2 with the given values. I could have also used $-1 + -11$ in the numerator since addition is commutative.
$M = \frac{-12}{2} = -6$	$\Rightarrow$	Simplified. -6 is halfway between -11 and -1.

Homework 1.7 Find the midpoint for the following pair of values.				
13) 4 and 24	14) 7 and -9	15) -17 and -2	16) 30 and -15	17) -15 and 15

1.7.4 Applying Formulas - Temperature

Fahrenheit is the temperature scale named after physicist Daniel Fahrenheit. It's the official temperature scale for the United States. Celsius (centigrade) is the temperature scale named after astronomer Anders Celsius. It's the official scale for most of the rest of the world. The formula $F = \frac{9}{5}C + 32$ converts a Celsius temperature to its equivalent Fahrenheit temperature. The formula $C = \frac{5(F-32)}{9}$ converts a Fahrenheit temperature to its Celsius equivalent. Here's some practice using temperature formulas.

Practice 1.7.4 Applying Formulas – Temperature

Use the appropriate formula to answer the following questions.

a) Someone who's only familiar with Fahrenheit temperature is told a natural hot spring is 65º Celsius. Would it be a good idea to soak in the pool made by the hot spring?		
$F = \frac{9}{5}C + 32$	$\Rightarrow$	Since I know the Celsius temperature, and I'm looking for the Fahrenheit temperature, I need the formula where I supply a value for *C* and simplify to find a value for *F*.
$F = \frac{9}{5}(65) + 32$	$\Rightarrow$	I substituted the given Celsius value into the formula.
$F = 149$	$\Rightarrow$	Then I simplified using the order of operations.
It's not a good idea since the temperature is too hot.	$\Rightarrow$	Answered the question in English. A temperature of 149° Fahrenheit can cause third degree burns in about 2 seconds.
b) Water freezes at 32º Fahrenheit. At what Celsius temperature does water freeze.		
$C = \dfrac{5(F-32)}{9}$	$\Rightarrow$	Since I've been given a value for *F*, and need to find a value for *C*, I'll use the formula where I supply the Fahrenheit temperature and find the equivalent Celsius temperature.
$C = \dfrac{5(32-32)}{9}$ $C = \dfrac{5(0)}{9} = 0$	$\Rightarrow$	I substituted the Fahrenheit value into the formula and simplified.
Water freezes at 0° Celsius.	$\Rightarrow$	I made sure to answer the question.

Homework 1.7 Use the appropriate formula to answer the following questions.

18) Water boils at 212º F. Find the Celsius temperature at which water boils.

19) A passenger from Canada says they prefer the temperature in your car to be around 20 degrees. To accommodate their needs what Fahrenheit temperature should you use?

20) On a trip to Europe you plan to cook a dish for your relatives, and you need to set the oven to 350º Fahrenheit. What Celsius temperature should you use?

21) 98.6º Fahrenheit is often used as the "normal" temperature for a person. What Celsius temperature would be considered normal?

22) At 48º Celsius it takes 5 minutes to develop a third-degree burn. At 60º Celsius it takes 5 seconds. Find the equivalent temperatures in degrees Fahrenheit.

23) With a car, it's important to never remove a radiator cap while the engine is hot. An American automobile manufacturer currently has a sticker on the radiator warning that, when hot, the antifreeze is between 195º and 220º Fahrenheit. If they plan on selling the car in Mexico, what Celsius temperatures should the warning be changed to? Round your temperatures to the nearest whole degree.

1.7.5 Applying Formulas - Paying Off a Loan

The function $A = 6{,}500 - 250t$ predicts the amount you'll still owe on a $6,500 loan, *A*, if you supply the number of months you've been paying, *t*. The function $t = 26 - 0.004A$ estimates how many months you've been paying on the loan, *t*, if you supply *A*, the amount still outstanding on the loan.

Homework 1.7 Use the appropriate function to answer the following questions.
24) You've been paying on the loan for 20 months, what is the outstanding balance?
25) How long will it take to get the loan down to around $1,000?
26) Your loan has dropped to $3,750, how many months have you been paying?
27) How much will you still owe on the loan after paying for 1 year?
28) What was the original amount of the loan?
29) How long will it take to pay off the loan?

Homework 1.7 Answers
1) Both expressions simplify to 46 so 11 is the solution.
2) Both expressions simplify to -9 so -2 is the solution.
3) The left expression simplifies to -6 and the right expression to 6. 1 is not the solution.
4) Both expressions simplify to $5/15$ or $1/3$ so 1 is the solution.
5) Both expressions simplify to $9/6$ or $3/2$ so 5 is the solution.
6) Both expressions simplify to $-32/36$ or $-8/9$ so 4 is the solution.
7) The left expression simplifies to 5 and the right expression to 6. 2 is not the solution.
8) Both expressions simplify to -16 so 12 is the solution.
9) Both expressions simplify to $52/36$ or $13/9$ so -20 is the solution.
10) Both expressions simplify to -9 so -4 is the solution.
11) Both expressions simplify to -15 so -10 is the solution.
12) The left expression simplifies to $-7/4$ and the right expression to 0. -1 is not the solution.

13) 14 14) -1 15) -9.5 16) 7.5 17) 0

18) $C = \frac{5(212 - 32)}{9} = 100$ Water boils at 100° Celsius.

19) $F = 9/5(20) + 32 = 68$ You should set the temperature at 68° Fahrenheit.

20) $C = \frac{5(350 - 32)}{9} = 176.\overline{6}$ The oven should be set to 177° Celsius.

21) $C = \frac{5(98.6 - 32)}{9} = 37$ Normal temperature would be 37° Celsius.

22) $F = 9/5(48) + 32 = 118.4$ $F = 9/5(60) + 32 = 140$ At 118° F it takes 48 seconds to develop a third-degree burn while at 140° F, it only takes about 5 seconds.

23) $C = \frac{5(195 - 32)}{9} \approx 90.6$, $C = \frac{5(220 - 32)}{9} \approx 104.4$ The sticker should warn that the temperature is between 90 and 104 degrees Celsius.

24) $A = 6{,}500 - 250(20) = 1{,}500$ After 20 months the outstanding balance is $1,500.

25) $t = 26 - 0.004(1{,}000) = 22$ It will take 22 months to reduce the loan amount to $1,000.

26) $t = 26 - 0.004(3{,}750) = 11$ It's taken 11 months for the loan to drop to $3,750.

27) $A = 6{,}500 - 250(12) = 3{,}500$ After paying on the loan for a year you'll still owe $3,500.

28) $A = 6{,}500 - 250(0) = 6{,}500$ At time 0 the original loan amount was $6,500.

29) $t = 26 - 0.004(0) = 26$, The loan amount will go to 0 in 26 months.

1.8 An Introduction to the Two-Column Table

So far, we've spent most of our time in this course simplifying expressions. Today we'll begin solving one-variable linear equations. The words, "one-variable", tell us that every variable term in our equation will have the same letter. The word "linear" tells us that the exponent on any variable will be a 1. (We'll discuss exponents in the next chapter.) In practice, people tend not to include "one-variable", and just use linear equation. An important point about linear equations is that, if there is a solution, there is usually only one.

1.8.1 Solving Linear Equations Using a Two-Column Table

The two-column table is based on the idea of using inverse operations to "undo" the operations happening to *x*. When undoing more than one operation, the two-column table will help you perform inverse operations in the right order.

Procedure **– Solving Equations Using a Two-Column Table**
1) Build a table with two columns and one row. Label the upper left cell, "Operations" and the upper right cell, "Inverses". (In time Oper and Inv are often used.) 2) Make a new blank row for each operation that is "affecting" the variable. 3) Begin in the left column and, imagining you're going to check an answer, record row by row the operations you would follow to simplify the expression. 4) In the right column list the inverse operations for the operations found in step 3. 5) To solve the equation, start at the <u>bottom</u> of the right column and work your way to the top, one row at a time, using the property of equality to "undo" operations. 6) Check any answers. If needed, the left column contains the correct order.

For example, to solve the linear equation $-3x+11=5$ I'd start by building a two-column table with one row for my labels.

Operations	Inverses

Next, I'd move to step 2 in the procedure and since there's two operations affecting *x* (multiplication by -3 and addition of 11) I'd build two blank rows.

Operations	Inverses

Moving to step 3 I'd take the point of view that I've replaced *x* with a number and I'm checking to see if it's a solution. By the order of operations, I'd multiply by -3 first and add 11 second so I fill in the left column with these operations. (The $\times$ symbol is for multiplication.)

Operations	Inverses
$\times -3$	
$+11$	

Following step 4, I'll move to the right column and list the inverse operation for each operation on the left. The inverse for +11 is -11 and the inverse for $\times -3$ is $\div -3$.

Operations	Inverses
$\times -3$	$\div -3$
$+11$	-11

To solve the equation, I'll start at the bottom of the right column, subtract 11 from both expressions and simplify.

Operations	Inverses
$\times -3$	$\div -3$
$+11$	-11

$$-3x+11=5$$
$$-3x+11-11=5-11$$
$$-3x=-6$$

Moving up a row, I'll divide both expressions by -3 and simplify. My solution is 2.

Operations	Inverses
$\times -3$	$\div -3$
$+11$	-11

$$\frac{-3x}{-3}=\frac{-6}{-3}$$
$$x=2$$

To check, I replaced x with my answer and, following the operations in the left column top to bottom, I simplified the left expression.

$$-3(2)+11 \quad 5$$
$$-6+11$$
$$5$$

Here's some practice solving linear equations using a two-column table.

Practice 1.8.1 Solving Linear Equations Using a Two-Column Table

Solve using a two-column table. Make sure to check your answer.

a) $\frac{x}{7}+5=9$

Operations	Inverses
$\div 7$	$\times 7$
$+5$	-5

$\Rightarrow$ There are two operators affecting x (division and addition) so my table needs two rows. If x was a number, I'd divide by 7 first and then adding 5. I listed these operators in the left column. In the right column I listed the inverse operators.

$$\frac{x}{7}+5-5=9-5$$
$$\frac{x}{7}=4$$

$\Rightarrow$ I started at the bottom of the right column, subtracted 5 from both expressions and simplified.

$$\frac{7}{1}\left(\frac{x}{7}\right)=4(7)$$
$$x=28$$

$\Rightarrow$ I moved up one row in the right column, multiplied both expressions by 7 and simplified.

$$\frac{28}{7}+5 \quad 9$$
$$4+5$$
$$9$$

$\Rightarrow$ I checked the answer. I noticed that the steps to simplify the left expression are exactly those in the left column from top to bottom.

Homework 1.8 Solve using a two-column table. Make sure to check your answer.

1) $6k-11=-5$ 2) $46=2L+24$ 3) $11+\frac{x}{6}=15$

4) $-8x-18=30$ 5) $\frac{y}{4}-8=-2$ 6) $\frac{m}{2}-6=-5$

1.8.2 Solving Linear Equations that Involve Grouping

Operations that are grouped, either explicitly or implicitly, are done first when you simplify an expression. That means they will be undone last when you build your two-column table.

Practice 1.8.2 Solving Linear Equations that Involve Grouping

Solve using a two-column table and check all solutions.

a) $\frac{m+3}{2}-6=-5$

Oper	Inv
$+3$	-3
$\div 2$	$\times 2$
-6	$+6$

$\Rightarrow$ If x was a value, I'd add 3 first (because of implicit grouping), divide by 2 second and then subtract 6. The right column contains the inverse operations.

$$\frac{m+3}{2}-6+6=-5+6$$
$$\frac{m+3}{2}=1$$

$\Rightarrow$ I started at the bottom of the right column, added 6 to both expressions and simplified.

$$\frac{2}{1}\left(\frac{m+3}{2}\right)=1(2)$$
$$m+3=2$$

$\Rightarrow$ I moved up a row, multiplied both sides by 2 and simplified.

$$m+3-3=2-3$$
$$m=-1$$

I moved up a row, subtracted 3 from both sides and simplified.

$$\frac{-1+3}{2}-6 \qquad -5$$
$$\frac{2}{2}-6$$
$$1-6$$
$$-5$$

$\Rightarrow$ I returned to the original equation and substituted my answer. After simplifying the left expression, the check looks good.

b) $2(x-1)+4=12$

Oper	Inv
-1	$+1$
$\times 2$	$\div 2$
$+4$	-4

$\Rightarrow$ If x was a value, I'd subtract 1 first (because of the parentheses), multiply by 2 second and then add 4. The right column contains the inverse operations.

$$2(x-1)+4-4=12-4$$
$$2(x-1)=8$$

$\Rightarrow$ I started at the bottom of the right column, subtracted 4 from both sides and simplified.

$$\frac{2(x-1)}{2}=\frac{8}{2}$$
$$x-1=4$$

$\Rightarrow$ I moved up a row, divided both expressions by 2 and simplified. The parentheses are no longer necessary.

$$x-1+1=4+1$$
$$x=5$$

$\Rightarrow$ I moved up a row, added 1 to both expressions and simplified. My answer is 5.

Homework 1.8 Solve using a two-column table and check all solutions.

7) $-3(y+4)+6=-12$	8) $-16=8(2x+1)$	9) $\frac{y+70}{3}-19=6$
10) $\frac{15+n}{4}+11=6$	11) $17=2+3(y+4)$	12) $1=\frac{x+12}{4}-9$

1.8.3 Solving More Complicated Linear Equations

Not let's include a few more operations in our equations.

Practice 1.8.3 Solving More Complicated Linear Equations

Solve using a two-column table and check all solutions.

a) $\frac{3(2x-7)}{5}-8=-5$

Oper	Inv
$\times 2$	$\div 2$
-7	$+7$
$\times 3$	$\div 3$
$\div 5$	$\times 5$
-8	$+8$

$\Rightarrow$ If x was a number, I'd (because of the parentheses) multiply by 2 first and subtract 7 second . Next, I'd multiply by 3 (due to implicit grouping), then divide by 5 and finally subtract 8. The right column contains the inverse operations.

$$\frac{3(2x-7)}{5}-8+8=-5+8$$
$$\frac{3(2x-7)}{5}=3$$

$\Rightarrow$ I started at the bottom of the right column, added 8 to both expressions and simplified.

$$\frac{5}{1}\left(\frac{3(2x-7)}{5}\right)=5(3)$$
$$3(2x-7)=15$$

$\Rightarrow$ I moved up a row, multiplied both sides by 5 and simplified.

$$\frac{3(2x-7)}{3}=\frac{15}{3}$$
$$2x-7=5$$

$\Rightarrow$ I moved up a row, divided both expressions by 3 and simplified.

$$2x-7+7=5+7$$
$$2x=12$$

$\Rightarrow$ I moved up a row, added 7 to both sides and simplified.

$$\frac{2x}{2}=\frac{12}{2}$$
$$x=6$$

$\Rightarrow$ I moved up a row, divided both expressions by 2 and simplified. My answer is 6.

Homework 1.8 Solve using a two-column table and check all solutions.

13) $-16 = 3(2x+1)-7$

14) $\dfrac{2(4x+7)}{3} = -6$

15) $\dfrac{4y+21}{5} - 3 = 6$

16) $15 = -4\left(\dfrac{3x}{5}+2\right)-1$

17) $\dfrac{17(7k-16)}{2} + 7 = -10$

18) $\dfrac{2(-3n+15)}{9} + 1 = -3$

19) $6 = \dfrac{9\left(\dfrac{2k}{5}+6\right)}{2} - 12$

20) $\dfrac{3(-4n-11)+6}{7} + 1 = -2$

21) $\dfrac{\dfrac{-2(-5n+6)}{3} - 128}{12} + 5 = -5$

Homework 1.8 Answers

1)

Oper	Inv
$\times 6$	$\div 6$
-11	$+11$

$k = 1$

2)

Oper	Inv
$\times 2$	$\div 2$
$+24$	-24

$L = 11$

3)

Oper	Inv
$\div 6$	$\times 6$
$+11$	-11

$x = 24$

4)

Oper	Inv
$\times -8$	$\div -8$
-18	$+18$

$x = -6$

5)

Oper	Inv
$\div 4$	$\times 4$
-8	$+8$

$y = 24$

6)

Oper	Inv
$\div 2$	$\times 2$
-6	$+6$

$m = 2$

7)

Oper	Inv
$+4$	-4
$\times -3$	$\div -3$
$+6$	-6

$y = 2$

8)

Oper	Inv
$\times 2$	$\div 2$
$+1$	-1
$\times 8$	$\div 8$

$x = -3/2$

9)

Oper	Inv
$+70$	-70
$\div 3$	$\times 3$
-19	$+19$

$y = 5$

10)

Oper	Inv
$+15$	-15
$\div 4$	$\times 4$
$+11$	-11

$n = -35$

11)

Oper	Inv
$+4$	-4
$\times 3$	$\div 3$
$+2$	-2

$y = 1$

12)

Oper	Inv
$+12$	-12
$\div 4$	$\times 4$
-9	$+9$

$x = 28$

13)

Oper	Inv
$\times 2$	$\div 2$
$+1$	-1
$\times 3$	$\div 3$
-7	$+7$

$x = -2$

14)

Oper	Inv
$\times 4$	$\div 4$
$+7$	-7
$\times 2$	$\div 2$
$\div 3$	$\times 3$

$x = -4$

15)

Oper	Inv
$\times 4$	$\div 4$
$+21$	-21
$\div 5$	$\times 5$
-3	$+3$

$y = 6$

16)

Oper	Inv
$\times 3$	$\div 3$
$\div 5$	$\times 5$
$+2$	-2
$\times -4$	$\div -4$
-1	$+1$

$x = -10$

17)

Oper	Inv
$\times 7$	$\div 7$
-16	$+16$
$\times 17$	$\div 17$
$\div 2$	$\times 2$
$+7$	-7

$k = 2$

18)

Oper	Inv
$\times -3$	$\div -3$
$+15$	-15
$\times 2$	$\div 2$
$\div 9$	$\times 9$
$+1$	-1

$n = 11$

19)

Oper	Inv
$\times 2$	$\div 2$
$\div 5$	$\times 5$
$+6$	-6
$\times 9$	$\div 9$
$\div 2$	$\times 2$
-12	$+12$

$k = -5$

20)

Oper	Inv
$\times -4$	$\div -4$
-11	$+11$
$\times 3$	$\div 3$
$+6$	-6
$\div 7$	$\times 7$
$+1$	-1

$n = -1/2$

21)

Oper	Inv
$\times -5$	$\div -5$
$+6$	-6
$\times -2$	$\div -2$
$\div 3$	$\times 3$
-128	$+128$
$\div 12$	$\times 12$
$+5$	-5

$n = 18/5$